

On a Simultaneous Approach to the Even and Odd Truncated Matricial Stieltjes Moment Problem

II. An α -Schur-Stieltjes-type algorithm for sequences of holomorphic matrix-valued functions

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The main goal of this paper is to achieve a simultaneous treatment of the even and odd truncated matricial Stieltjes moment problems in the most general case. These results are generalizations of results of Chen and Hu [5, 17] which considered the particular case $\alpha = 0$. Our approach is based on Schur analysis methods. More precisely, we use two interrelated versions of Schur-type algorithms, namely an algebraic one and a function-theoretic one. The algebraic version was worked out in a former paper of the authors. It is an algorithm which is applied to finite or infinite sequences of complex matrices. The construction and investigation of the function-theoretic version of our Schur-type algorithm is a central theme of this paper. This algorithm will be applied to relevant subclasses of holomorphic matrix-valued functions of the Stieltjes class. Using recent results on the holomorphicity of the Moore-Penrose inverse of matrix-valued Stieltjes functions, we obtain a complete description of the solution set of the moment problem under consideration in the most general situation.

1. Introduction

This paper is a continuation of [12]. The main goal is to achieve a simultaneous treatment of the even and odd cases of a truncated matricial Stieltjes moment problem. We are guided by our investigations on the truncated matricial Hamburger moment problem in [11], where we simultaneously treated the even and odd cases. Our approach in [11] is based on Schur analysis methods the origin of which goes back to the fundamental memoir [22] by R. Nevanlinna. We used two interrelated versions of Schur-type algorithms, namely an algebraic one and a function-theoretic one. The algebraic version had been worked out in the paper [14], whereas the function-theoretic version and the interplay

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of both versions were investigated in [11]. Against to this background, we want to treat now simultaneously the even and odd cases for an analogous truncated matricial Stieltjes problem. More precisely, our approach will be based again on Schur analysis methods, namely on the interplay between two versions of Schur-type algorithms, an algebraic one and a function-theoretic one. The investigation of the algebraic version of our Schur-type algorithm was the main content of the paper [12]. The first central theme of this paper is to work out the function-theoretic version of our Schur-type algorithm. A careful analysis of the interplay between both versions will lead us finally to a complete solution of the original truncated matricial Stieltjes problem in the most general case. This is the main achievement of this paper.

Though, roughly speaking, we repeat the basic strategy used in [11, 14] the concrete constructions are rather different. What concerns the algebraic versions of our Schur-type algorithm, these differences were described in great detail in [12]. Roughly speaking, the main difference can be characterized as follows: In the Hamburger case the algebraic version of our Schur-type algorithm is a two-step algorithm, whereas in the Stieltjes case we are lead to a one-step algorithm. This observation reflects the nature of finite Hankel non-negative definite sequences from $\mathbb{C}^{q \times q}$ on the one side and finite α -Stieltjes non-negative definite sequences from $\mathbb{C}^{q \times q}$ on the other side. Using matricial versions of Hamburger-Nevanlinna theorems in [11], we reformulated the truncated matricial Hamburger moment problem under study into an equivalent problem of prescribing an asymptotic expression for matrix-valued functions which are holomorphic in the open upper half plane. Guided by these experiences, we prove now Stieltjes-type versions of Hamburger-Nevanlinna-type theorems, which enable us to reformulate the original truncated matricial Stieltjes moment problem under study into an equivalent problem of prescribing an asymptotic expansion for a special class of matrix-valued functions which are holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$ where α is a given real number. More precisely, this special class is the class $\mathcal{S}_{q, [\alpha, +\infty)}$ of all $[\alpha, +\infty)$ -Stieltjes function of order q , which is introduced in Definition 2.1. We frequently apply results from our former paper [13], which contains a detailed treatment of the class $\mathcal{S}_{q, [\alpha, +\infty)}$ and some important subclasses. The construction and investigation of a Schur-type algorithm for $q \times q$ matrix-valued functions holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$ which preserves special subclasses of $\mathcal{S}_{q, [\alpha, +\infty)}$ is a central theme of this paper. The idea how to build the algorithm in the matrix case was inspired by some constructions in the papers [5, 17] by Chen and Hu. An essential point of our approach is an intensive use of the interplay between the function-theoretic and algebraic versions of our matricial Schur-type algorithms. Both algorithms are formulated in terms of Moore-Penrose inverses of matrices. What concerns the function-theoretic version, it can be said that its effectiveness is mostly caused by recent results from [13] on the holomorphicity of the Moore-Penrose inverse of special classes of holomorphic matrix-valued functions.

In order to describe more concretely the central topics studied in this paper, we introduce some notation. Throughout this paper, let p and q be positive integers. Let \mathbb{N} , \mathbb{N}_0 , \mathbb{Z} , \mathbb{R} , and \mathbb{C} be the set of all positive integers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers, respectively. For every choice of $\rho, \kappa \in \mathbb{R} \cup \{-\infty, +\infty\}$, let $\mathbb{Z}_{\rho, \kappa} := \{k \in \mathbb{Z} \mid \rho \leq k \leq \kappa\}$.

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We will write $\mathbb{C}^{p \times q}$, $\mathbb{C}_H^{q \times q}$, $\mathbb{C}_{\geq}^{q \times q}$, and $\mathbb{C}_{>}^{q \times q}$ for the set of all complex $p \times q$ matrices, the set of all Hermitian complex $q \times q$ matrices, the set of all non-negative Hermitian complex $q \times q$ matrices, and the set of all positive Hermitian complex $q \times q$ matrices, respectively.

We will use $\mathfrak{B}_{\mathbb{R}}$ to denote the σ -algebra of all Borel subsets of \mathbb{R} . Let $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$. Then, let $\mathfrak{B}_{\Omega} := \mathfrak{B}_{\mathbb{R}} \cap \Omega$. Furthermore, we will write $\mathcal{M}_{\geq}^q(\Omega)$ to designate the set of all non-negative Hermitian $q \times q$ measures defined on \mathfrak{B}_{Ω} , i. e., the set of all σ -additive mappings $\mu: \mathfrak{B}_{\Omega} \rightarrow \mathbb{C}_{\geq}^{q \times q}$. We will use the integration theory with respect to non-negative Hermitian $q \times q$ measures, which was worked out independently by I. S. Kats [18] and M. Rosenberg [25]. Some features of this theory are sketched in Appendix A. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we will use $\mathcal{M}_{q,\kappa}^{\geq}(\Omega)$ to denote the set of all $\sigma \in \mathcal{M}_{\geq}^q(\Omega)$ such that the integral

$$s_j^{(\sigma)} := \int_{\Omega} x^j \sigma(dx) \quad (1.1)$$

exists for all $j \in \mathbb{Z}_{0,\kappa}$.

Remark 1.1. If $k, \ell \in \mathbb{N}_0$ with $k < \ell$, then the inclusion $\mathcal{M}_{q,\ell}^{\geq}(\Omega) \subseteq \mathcal{M}_{q,k}^{\geq}(\Omega)$ holds true.

The central problem in this paper is the truncated version of the following power moment problem of Stieltjes-type

Problem ($\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^{\kappa}, =]$). Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_{q,\kappa}^{\geq}[[\alpha, +\infty); (s_j)_{j=0}^{\kappa}, =]$ of all $\sigma \in \mathcal{M}_{q,\kappa}^{\geq}([\alpha, +\infty))$ for which $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,\kappa}$.

We note that there is a further matricial version of the truncated Stieltjes moment problem:

Problem ($\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^m, \leq]$). Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_{q,m}^{\geq}[[\alpha, +\infty); (s_j)_{j=0}^m, \leq]$ of all $\sigma \in \mathcal{M}_{q,m}^{\geq}([\alpha, +\infty))$ for which $s_j^{(\sigma)} = s_j$ is satisfied for each $j \in \mathbb{Z}_{0,m-1}$, whereas the matrix $s_m - s_m^{(\sigma)}$ is non-negative Hermitian.

A detailed treatment of the history of these two moment problems is contained in the introduction to the paper [8].

In order to state a necessary and sufficient condition for the solvability of each of the above formulated moment problems, we have to recall the notion of two types of sequences of matrices. If $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n}$ is a sequence of complex $q \times q$ matrices, then $(s_j)_{j=0}^{2n}$ is called *Hankel non-negative definite* if the block Hankel matrix

$$H_n := [s_{j+k}]_{j,k=0}^n$$

is non-negative Hermitian. Let $(s_j)_{j=0}^{\infty}$ be a sequence of complex $q \times q$ matrices. Then $(s_j)_{j=0}^{\infty}$ is called *Hankel non-negative definite* if $(s_j)_{j=0}^{2n}$ is Hankel non-negative definite for all $n \in \mathbb{N}_0$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we will write $\mathcal{H}_{q,2\kappa}^{\geq}$ for the set of all Hankel non-negative definite sequences $(s_j)_{j=0}^{2\kappa}$ of complex $q \times q$ matrices. Furthermore, for all

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$n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\geq,e}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there exist complex $q \times q$ matrices s_{2n+1} and s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^{\geq}$, whereas $\mathcal{H}_{q,2n+1}^{\geq,e}$ stands for the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there exist some $s_{2n+2} \in \mathbb{C}^{q \times q}$ such that $(s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^{\geq}$. For each $m \in \mathbb{N}_0$, the elements of the set $\mathcal{H}_{q,m}^{\geq,e}$ are called *Hankel non-negative definite extendable sequences*. For technical reason, we set $\mathcal{H}_{q,\infty}^{\geq,e} := \mathcal{H}_{q,\infty}^{\geq}$.

Besides the just introduced classes of sequences of complex $q \times q$ matrices we need analogous classes of sequences of complex $q \times q$ matrices which take into account the influence of the prescribed number $\alpha \in \mathbb{R}$: Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then, for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, we introduce the block Hankel matrix

$$K_n := [s_{j+k+1}]_{j,k=0}^n.$$

Let $\alpha \in \mathbb{R}$. Now we will introduce several classes of finite or infinite sequences of complex $q \times q$ matrices which are characterized by the sequences $(s_j)_{j=0}^{\kappa}$ and $(-\alpha s_j + s_{j+1})_{j=0}^{\kappa-1}$. Let $\mathcal{K}_{q,0,\alpha}^{\geq} := \mathcal{H}_{q,0}^{\geq}$, and, for all $n \in \mathbb{N}$, let $\mathcal{K}_{q,2n,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_{n-1} + K_{n-1}$ are both non-negative Hermitian, i. e.,

$$\mathcal{K}_{q,2n,\alpha}^{\geq} = \left\{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \mid (-\alpha s_j + s_{j+1})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq} \right\}. \quad (1.2)$$

Furthermore, for all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_n + K_n$ are both non-negative Hermitian, i. e.,

$$\mathcal{K}_{q,2n+1,\alpha}^{\geq} := \left\{ (s_j)_{j=0}^{2n+1} \text{ from } \mathbb{C}^{q \times q} \mid \left\{ (s_j)_{j=0}^{2n}, (-\alpha s_j + s_{j+1})_{j=0}^{2n} \right\} \subseteq \mathcal{H}_{q,2n}^{\geq} \right\}. \quad (1.3)$$

Let $\mathcal{K}_{q,\infty,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ for all $m \in \mathbb{N}_0$. Formulas (1.2) and (1.3) show that the sets $\mathcal{K}_{q,2n,\alpha}^{\geq}$ and $\mathcal{K}_{q,2n+1,\alpha}^{\geq}$ are determined by two conditions. The condition $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ ensures that a particular Hamburger moment problem associated with the sequence $(s_j)_{j=0}^{2n}$ is solvable (see, e. g. [8, Theorem 4.16]). The second condition $(-\alpha s_j + s_{j+1})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq}$ (resp. $(-\alpha s_j + s_{j+1})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$) controls that the original sequences $(s_j)_{j=0}^{2n}$ and $(s_j)_{j=0}^{2n+1}$ are well adapted to the interval $[\alpha, +\infty)$. Let $m \in \mathbb{N}_0$. Then, let $\mathcal{K}_{q,m,\alpha}^{\geq,e}$ be the set of all sequences $(s_j)_{j=0}^m$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{m+1} such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1,\alpha}^{\geq}$. Further, let $\mathcal{K}_{q,\infty,\alpha}^{\geq,e} := \mathcal{K}_{q,\infty,\alpha}^{\geq}$. We call a sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices *α -Stieltjes non-negative definite* if it belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$ and *α -Stieltjes non-negative definite extendable* if it belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$.

Now we can characterize the situations in which the mentioned problems have a solution:

Theorem 1.2 ([7, Theorems 1.3 and 1.4]). *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then:*

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- (a) $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$.
- (b) $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq}$.

In the case $\alpha = 0$ there is some important work of Bolotnikov [2] and Chen/Hu [5, 17] which influenced our investigations. This will be explained now in more detail. Theorem 1.2(a) was proved in Bolotnikov [17, Lemma 1.7]. Bolotnikov also stated a parametrization of the solution set of Problem $M[[0, +\infty); (s_j)_{j=0}^m, \leq]$ in the language of the Stieltjes transforms (see [17, Theorems 5.3 and 6.6]). Chen and Hu stated a parametrization of the solution set of Problem $M[[0, +\infty); (s_j)_{j=0}^m, =]$ in the language of the Stieltjes transforms (see [17, Theorem 4.1]).

The main goal of our paper here is to give via Stieltjes transformation a parametrization of the solution set of Problem $M[[\alpha, +\infty); (s_j)_{j=0}^m, =]$ (see Theorems 13.1 and 13.4).

2. The class $\mathcal{S}_{q, [\alpha, +\infty)}$

The use of several classes of holomorphic matrix-valued functions is one of the special features of this paper. In this section, we summarize some basic facts about the class of $[\alpha, +\infty)$ -Stieltjes functions of order q , which are mostly taken from our former paper [13]. If $A \in \mathbb{C}^{q \times q}$, then let $\operatorname{Re} A := \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A := \frac{1}{2i}(A - A^*)$ be the real part and the imaginary part of A , respectively. Let $\Pi_+ := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the open upper half plane of \mathbb{C} . The first class of functions which plays an essential role in this paper, is the following.

Definition 2.1. Let $\alpha \in \mathbb{R}$ and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$. Then F is called a $[\alpha, +\infty)$ -Stieltjes function of order q if F satisfies the following three conditions:

- (I) F is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$.
- (II) For all $w \in \Pi_+$, the matrix $\operatorname{Im}[F(w)]$ is non-negative Hermitian.
- (III) For all $w \in (-\infty, \alpha)$, the matrix $F(w)$ is non-negative Hermitian.

We denote by $\mathcal{S}_{q, [\alpha, +\infty)}$ the set of all $[\alpha, +\infty)$ -Stieltjes functions of order q .

For a comprehensive survey on the class $\mathcal{S}_{q, [\alpha, +\infty)}$, we refer the reader to [13]. The functions belonging to the class $\mathcal{S}_{q, [\alpha, +\infty)}$ admit an important integral representation. To state this, we introduce some terminology: If μ is a non-negative Hermitian $q \times q$ measure on a measurable space (Ω, \mathfrak{A}) , then we will use $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ to denote the space of all Borel-measurable functions $f: \Omega \rightarrow \mathbb{K}$ for which the integral $\int_{\Omega} f d\mu$ exists. In preparing the desired integral representation, we observe that, for all $\mu \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ and each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, the function $h_{\alpha, z}: [\alpha, +\infty) \rightarrow \mathbb{C}$ defined by $h_{\alpha, z}(t) := (1 + t - \alpha)/(t - z)$ belongs to $\mathcal{L}^1([\alpha, +\infty), \mathfrak{B}_{[\alpha, +\infty)}, \mu; \mathbb{C})$.

Theorem 2.2 (cf. [13, Theorem 3.6]). *Let $\alpha \in \mathbb{R}$ and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$. Then:*

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- (a) If $F \in \mathcal{S}_{q, [\alpha, +\infty)}$, then there are a unique matrix $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and a unique non-negative Hermitian measure $\mu \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ such that

$$F(z) = \gamma + \int_{[\alpha, +\infty)} \frac{1+t-\alpha}{t-z} \mu(dt) \quad (2.1)$$

holds true for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

- (b) If there are a matrix $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and a non-negative Hermitian measure $\mu \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ such that F can be represented via (2.1) for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then F belongs to the class $\mathcal{S}_{q, [\alpha, +\infty)}$.

For all $F \in \mathcal{S}_{q, [\alpha, +\infty)}$, we will write (γ_F, μ_F) for the unique pair $(\gamma, \mu) \in \mathbb{C}_{\geq}^{q \times q} \times \mathcal{M}_{\geq}^q([\alpha, +\infty))$ for which the representation (2.1) holds true for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

In the special case that $q = 1$ and $\alpha = 0$ Theorem 2.2 can be found in Krein/Nudelman [21, Appendix].

Remark 2.3 ([13, Remark 3.12]). Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q, [\alpha, +\infty)}$. Further let $A \in \mathbb{C}^{q \times q}$ be such that $\gamma_F + A$ is non-negative Hermitian. Then $G := F + A \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $(\gamma_G, \mu_G) = (\gamma_F + A, \mu_F)$.

Proposition 2.4 ([13, Proposition 3.13]). Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q, [\alpha, +\infty)}$. Then $\lim_{y \rightarrow +\infty} F(iy) = \gamma_F$.

We are particularly interested in the structure of the values of functions belonging to $\mathcal{S}_{q, [\alpha, +\infty)}$. For this reason, we introduce some notation: For all $A \in \mathbb{C}^{p \times q}$, let $\mathcal{N}(A)$ be the null space of A and $\mathcal{R}(A)$ be the column space of A .

Proposition 2.5 (cf. [13, Proposition 3.15]). Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q, [\alpha, +\infty)}$. For all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then

$$\mathcal{R}(F(z)) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty))) \quad \text{and} \quad \mathcal{N}(F(z)) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty))).$$

Proposition 2.5 contains essential information on the class $\mathcal{S}_{q, [\alpha, +\infty)}$. It indicates that, for an arbitrary function F belonging to $\mathcal{S}_{q, [\alpha, +\infty)}$, the null space $\mathcal{N}(F(z))$ and the column space $\mathcal{R}(F(z))$ are independent of the concrete point $z \in \mathbb{C} \setminus [\alpha, +\infty)$ and, furthermore, in which way these linear subspaces of \mathbb{C}^q are determined by the pair (γ_F, μ_F) of F .

In the sequel, we will sometimes meet situations where interrelations of the null space (resp. column space) of a function $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ to the null space (resp. column space) of a given matrix $A \in \mathbb{C}^{p \times q}$ are of interest. More precisely, we will frequently apply the following auxiliary result. In preparing its formulation we note that for a matrix $A \in \mathbb{C}^{p \times q}$ we denote by A^\dagger its Moore-Penrose inverse. This means A^\dagger is the unique matrix X from $\mathbb{C}^{q \times p}$ which satisfies the four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$.

Lemma 2.6 (cf. [13, Lemma 3.18]). Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{p \times q}$, and let $F \in \mathcal{S}_{q, [\alpha, +\infty)}$. Then the following statements are equivalent:

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- (i) $\mathcal{N}(A) \subseteq \mathcal{N}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (ii) There is a $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$ such that $\mathcal{N}(A) \subseteq \mathcal{N}(F(z_0))$.
- (iii) $\mathcal{N}(A) \subseteq \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty)))$.
- (iv) $FA^\dagger A = F$.
- (v) $\mathcal{R}(F(z)) \subseteq \mathcal{R}(A^*)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (vi) There is a $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$ such that $\mathcal{R}(F(z_0)) \subseteq \mathcal{R}(A^*)$.
- (vii) $\mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty))) \subseteq \mathcal{R}(A^*)$.
- (viii) $A^\dagger AF = F$.

A generic application of Lemma 2.6 will be concerned with situations where the matrix A even belongs to $\mathbb{C}_{\geq}^{q \times q}$.

In our subsequent considerations we will very often use the Moore-Penrose inverse of functions belonging to the class $\mathcal{S}_{q, [\alpha, +\infty)}$. In this connection, the following result turns out to be of central importance.

Proposition 2.7 ([13, Theorem 6.3]). *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q, [\alpha, +\infty)}$. Then $G: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ defined by $G(z) := -(z - \alpha)^{-1}[F(z)]^\dagger$ belongs to the class $\mathcal{S}_{q, [\alpha, +\infty)}$ as well.*

3. On some subclasses of $\mathcal{S}_{q, [\alpha, +\infty)}$

An essential feature of our subsequent considerations is the use of several subclasses of $\mathcal{S}_{q, [\alpha, +\infty)}$. In this section, we summarize basic facts about these subclasses under the special orientation of this paper. These subclasses are characterized by growth properties on the positive imaginary axis. It should be mentioned that the scalar versions of the function classes were introduced and studied in the paper [19]. Next we recall the Euclidean norm of a matrix. For each $B \in \mathbb{C}^{q \times q}$, let $\text{tr } B$ be the trace of B . If $A \in \mathbb{C}^{p \times q}$, then $\|A\|_E := \sqrt{\text{tr}(A^*A)}$ is the Euclidean norm of A . An important subclass of the class $\mathcal{S}_{q, [\alpha, +\infty)}$ is the set

$$\mathcal{S}_{0, q, [\alpha, +\infty)} := \left\{ F \in \mathcal{S}_{q, [\alpha, +\infty)} \left| \sup_{y \in [1, +\infty)} y \|F(iy)\|_E < +\infty \right. \right\}. \quad (3.1)$$

Now we want to characterize the class $\mathcal{S}_{0, q, [\alpha, +\infty)}$ via an integral representation.

Lemma 3.1 ([13, Lemma A.4]). *Let Ω be a non-empty closed subset of \mathbb{R} and let $\sigma \in \mathcal{M}_{\geq}^q(\Omega)$. Then:*

- (a) *For each $w \in \mathbb{C} \setminus \Omega$, the function $g_w: \Omega \rightarrow \mathbb{C}$ defined by $g_w(t) := 1/(t - w)$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$.*

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(b) The matrix-valued function $S: \mathbb{C} \setminus \Omega \rightarrow \mathbb{C}^{q \times q}$ given by $S(w) := \int_{\Omega} g_w d\sigma$ satisfies $\mathcal{R}(S(z)) = \mathcal{R}(\sigma(\Omega))$ and in particular $\text{rank } S(z) = \text{rank } \sigma(\Omega)$ for each $z \in \mathbb{C} \setminus [\inf \Omega, \sup \Omega]$.

(c) $-\text{i} \lim_{y \rightarrow +\infty} y S(\text{i}y) = \sigma(\Omega)$.

Let Ω be a non-empty closed subset of \mathbb{R} and let $\sigma \in \mathcal{M}_{\geq}^q(\Omega)$. Then, in view of Lemma 3.1(a), the matrix-valued function $S_{\sigma}: \mathbb{C} \setminus \Omega \rightarrow \mathbb{C}^{q \times q}$ given by

$$S_{\sigma}(z) := \int_{\Omega} \frac{1}{t-z} \sigma(dt) \quad (3.2)$$

is well-defined and called Ω -Stieltjes transform of σ .

Let $z \in \mathbb{C} \setminus \Omega$. Then obviously $\bar{z} \in \mathbb{C} \setminus \Omega$ and from (3.2) it follows

$$S_{\sigma}(\bar{z}) = \int_{\Omega} \frac{1}{t-\bar{z}} \sigma(dt) = \int_{\Omega} \overline{\left(\frac{1}{t-z} \right)} \sigma(dt) = \left[\int_{\Omega} \frac{1}{t-z} \sigma(dt) \right]^* = [S_{\sigma}(z)]^*. \quad (3.3)$$

Now we consider the case $\Omega = [\alpha, +\infty)$. We want to characterize the set of all $[\alpha, +\infty)$ -Stieltjes transforms of measures belonging to $\mathcal{M}_{\geq}^q([\alpha, +\infty))$.

Theorem 3.2. *The mapping $\sigma \mapsto S_{\sigma}$ is a bijective correspondence between $\mathcal{M}_{\geq}^q([\alpha, +\infty))$ and $\mathcal{S}_{0,q, [\alpha, +\infty)}$. In particular $\mathcal{S}_{0,q, [\alpha, +\infty)} = \{S_{\sigma} \mid \sigma \in \mathcal{M}_{\geq}^q([\alpha, +\infty))\}$.*

Proof. This is an immediate consequence of [13, Theorem 5.1]. \square

For each $F \in \mathcal{S}_{0,q, [\alpha, +\infty)}$, the unique measure $\sigma \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ satisfying $S_{\sigma} = F$ is called the $[\alpha, +\infty)$ -Stieltjes measure of F . We will also write σ_F for σ . Theorem 3.2 indicates that the $[\alpha, +\infty)$ -Stieltjes transform S_{σ} of a measure $\sigma \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ is characterized by a particular mild growth on the positive imaginary axis.

Corollary 3.3. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{0,q, [\alpha, +\infty)}$. For $z \in \mathbb{C} \setminus [\alpha, +\infty)$ then $\bar{z} \in \mathbb{C} \setminus [\alpha, +\infty)$ and $F(\bar{z}) = [F(z)]^*$.*

Proof. Combine Theorem 3.2 and (3.3) \square

Proposition 3.4 ([13, Proposition 6.4]). *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{0,q, [\alpha, +\infty)}$, and let σ_F be the $[\alpha, +\infty)$ -Stieltjes measure of F . Then $G: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ defined by $G(z) := -(z - \alpha)^{-1} [F(z)]^{\dagger}$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}$ and*

$$\gamma_G = [\sigma_F([\alpha, +\infty))]^{\dagger}.$$

In particular, if F is not the constant function with value $O_{q \times q}$, then $G \in \mathcal{S}_{q, [\alpha, +\infty)} \setminus \mathcal{S}_{0,q, [\alpha, +\infty)}$.

In view of Theorem 3.2, Problem $\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^{\kappa}, =]$ can be given a first reformulation as an equivalent problem in the class $\mathcal{S}_{0,q, [\alpha, +\infty)}$ as follows:

3. On some subclasses of $\mathcal{S}_{q, [\alpha, +\infty)}$

Problem ($\mathcal{S}[[\alpha, +\infty); (s_j)_{j=0}^\kappa, =]$). Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$ of all $F \in \mathcal{S}_{0, q, [\alpha, +\infty)}$ the $[\alpha, +\infty)$ -Stieltjes measure of which belongs to $\mathcal{M}_{q, \kappa}^\geq[[\alpha, +\infty); (s_j)_{j=0}^\kappa, =]$.

In Section 6, we will state a reformulation of the original power moment problem $\mathcal{M}[[\alpha, +\infty); (s_j)_{j=0}^\kappa, =]$ as an equivalent problem of finding a prescribed asymptotic expansion in a sector of the open upper half plane Π_+ . Furthermore, we will see that a detailed analysis of the behavior on the positive imaginary axis of the concrete functions of $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ under study is extremely useful. For this reason, we turn now our attention to some subclasses of $\mathcal{S}_{q, [\alpha, +\infty)}$ which are described in terms of their growth on the positive imaginary axis. First we consider the set

$$\mathcal{S}_{q, [\alpha, +\infty)}^\diamond := \left\{ F \in \mathcal{S}_{q, [\alpha, +\infty)} \mid \lim_{y \rightarrow +\infty} \|F(iy)\|_E = 0 \right\}. \quad (3.4)$$

In the following considerations we associate with a function $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ often the unique ordered pair (γ_F, μ_F) given via Theorem 2.2(a).

Remark 3.5 ([13, Corollary 3.14]). Let $\alpha \in \mathbb{R}$. Then

$$\mathcal{S}_{q, [\alpha, +\infty)}^\diamond = \left\{ F \in \mathcal{S}_{q, [\alpha, +\infty)} \mid \gamma_F = O_{q \times q} \right\}.$$

Remark 3.6. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(p_k)_{k=1}^n$ be a sequence of positive integers. For all $k \in \mathbb{Z}_{1, n}$, let $F_k \in \mathcal{S}_{p_k, [\alpha, +\infty)}^\diamond$ and let $A_k \in \mathbb{C}^{p_k \times q}$. In view of Remark 3.5 and [13, Remark 3.11], then $\sum_{k=1}^n A_k^* F_k A_k \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$.

Now we state modifications of Proposition 2.5 and Lemma 2.6 for $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond$.

Proposition 3.7. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$, and let μ_F be given via Theorem 2.2(a). For all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then*

$$\mathcal{R}(F(z)) = \mathcal{R}(\mu_F([\alpha, +\infty))) \quad \text{and} \quad \mathcal{N}(F(z)) = \mathcal{N}(\mu_F([\alpha, +\infty))).$$

Proof. Combine Proposition 2.5 and Remark 3.5. □

Lemma 3.8. *Let $A \in \mathbb{C}^{p \times q}$ and let $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$. Then the statements*

$$(ix) \quad \mathcal{R}(\mu_F([\alpha, +\infty))) \subseteq \mathcal{R}(A^*).$$

and

$$(x) \quad \mathcal{N}(A) \subseteq \mathcal{N}(\mu_F([\alpha, +\infty))).$$

are equivalent. Furthermore, (ix) is equivalent to each of the statements (i)–(viii) stated in Lemma 2.6.

Proof. Combine Remark 3.5 and Lemma 2.6. □

For all $\alpha \in \mathbb{R}$ and all $\kappa \in \mathbb{N} \cup \{\infty\}$, we now consider the class

$$\mathcal{S}_{\kappa, q, [\alpha, +\infty)} := \left\{ F \in \mathcal{S}_{0, q, [\alpha, +\infty)} \mid \sigma_F \in \mathcal{M}_{q, \kappa}^\geq([\alpha, +\infty)) \right\}. \quad (3.5)$$

4. On the class $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A]$

Lemma 3.9. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}$ with $[\alpha, +\infty)$ -Stieltjes measure σ_F . For each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then*

$$\mathcal{R}(F(z)) = \mathcal{R}(\sigma_F([\alpha, +\infty))) \quad \text{and} \quad \mathcal{N}(F(z)) = \mathcal{N}(\sigma_F([\alpha, +\infty))).$$

Proof. This is an immediate consequence of [13, Proposition 5.3]. \square

Remark 3.10. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}$. Then from (3.5) and (3.1) we see that $\lim_{y \rightarrow +\infty} F(iy) = O_{q \times q}$.

Remark 3.11. Let $\alpha \in \mathbb{R}$. In view of the Remarks 1.1 and 3.10, then

$$\mathcal{S}_{\infty, q, [\alpha, +\infty)} \subseteq \mathcal{S}_{\ell, q, [\alpha, +\infty)} \subseteq \mathcal{S}_{m, q, [\alpha, +\infty)} \subseteq \mathcal{S}_{0, q, [\alpha, +\infty)} \subseteq \mathcal{S}_{q, [\alpha, +\infty)}^\diamond \subseteq \mathcal{S}_{q, [\alpha, +\infty)} \quad (3.6)$$

for all $\ell, m \in \mathbb{N}_0$ with $\ell \geq m$.

4. On the class $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A]$

In this section, we consider a particular subclass of the class $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond$, which was introduced in (3.4). We have seen in Proposition 2.5 that, for an arbitrary function $F \in \mathcal{S}_{q, [\alpha, +\infty)}$, the null space of $F(z)$ is independent from the concrete choice of $z \in \mathbb{C} \setminus [\alpha, +\infty)$. For the case $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$, a complete description of this constant null space was given in Proposition 3.7. Against to this background, we single out now a special subclass of $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond$, which is characterized by the interrelation of the constant null space to the null space of a prescribed matrix $A \in \mathbb{C}^{p \times q}$. More precisely, for all $A \in \mathbb{C}^{p \times q}$, let

$$\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A] := \left\{ F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond \mid \mathcal{N}(A) \subseteq \mathcal{N}(\mu_F([\alpha, +\infty))) \right\} \quad (4.1)$$

In our later investigations the role of the matrix A will be taken by matrices which are generated from the sequence of data matrices of the problem under consideration via a Schur-type algorithm.

Remark 4.1. Let $\alpha \in \mathbb{R}$. If $A \in \mathbb{C}^{p \times q}$ satisfies $\mathcal{N}(A) = \{O_{q \times 1}\}$, then $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A] = \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$. In particular, this situation arises in the case that $p = q$ and $\det A \neq 0$ are fulfilled.

In the following considerations, we associate with a function $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ often the unique ordered pair (γ_F, μ_F) given via Theorem 2.2(a).

Example 4.2. Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{p \times q}$, and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $F(z) := O_{q \times q}$. In view of [13, Example 3.19], then one can easily see that $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A]$ and μ_F is the zero measure belonging to $\mathcal{M}_{\geq}^q([\alpha, +\infty))$.

Lemma 4.3. *Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{p \times q}$ and $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$. Then the following statements are equivalent:*

$$(i) \quad F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A].$$

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- (ii) $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$ and $\mathcal{N}(A) \subseteq \mathcal{N}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (iii) $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$, $\lim_{y \rightarrow +\infty} \|F(iy)\|_E = 0$, and $\mathcal{N}(A) \subseteq \mathcal{N}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (iv) $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$, $\gamma_F = O_{q \times q}$, and $\mathcal{N}(A) \subseteq \mathcal{N}(\mu_F([\alpha, +\infty)))$.

Proof. This follows from (4.1), Lemma 3.8, (3.4), and Remark 3.5. \square

Remark 4.4. Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{p \times q}$, and let $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$. In view of Remark 3.5 and Lemma 2.6, then the following statements are equivalent:

- (i) $F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A]$.
- (ii) There is a $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$ such that $\mathcal{N}(A) \subseteq \mathcal{N}(F(z_0))$.
- (iii) $\mathcal{N}(A) \subseteq \mathcal{N}(\mu_F([\alpha, +\infty)))$.
- (iv) $FA^\dagger A = F$.
- (v) $\mathcal{R}(F(z)) \subseteq \mathcal{R}(A^*)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (vi) There is a $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$ such that $\mathcal{R}(F(z_0)) \subseteq \mathcal{R}(A^*)$.
- (vii) $\mathcal{R}(\mu_F([\alpha, +\infty))) \subseteq \mathcal{R}(A^*)$.
- (viii) $A^\dagger AF = F$.

Remark 4.5. Let $\alpha \in \mathbb{R}$, let $r \in \mathbb{N}$, and let $A \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{r \times q}$ be such that $\mathcal{N}(A) \subseteq \mathcal{N}(B)$. Then $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[B] \subseteq \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A]$.

Remark 4.6. Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{p \times q}$. In view of Remark 4.4, $A^\dagger A = (A^\dagger A)^*$ and [13, Remark 3.20], then $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A] = \{A^\dagger A F A^\dagger A : F \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond\}$.

The following result contains essential information on the structure of the set $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A]$, where $A \in \mathbb{C}^{p \times q}$.

Proposition 4.7. *Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{p \times q}$. Then:*

- (a) *If $A = O_{p \times q}$, then $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A] = \{F\}$, where $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ is defined by $F(z) := O_{q \times q}$.*
- (b) *Suppose that $r := \text{rank } A$ fulfills $r \geq 1$. Let u_1, u_2, \dots, u_r be an orthonormal basis of $\mathcal{R}(A^*)$ and let $U := [u_1, u_2, \dots, u_r]$. Then:*
 - (b1) $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A] = \{U f U^* \mid f \in \mathcal{S}_{r, [\alpha, +\infty)}^\diamond\}$.
 - (b2) *If $f, g \in \mathcal{S}_{r, [\alpha, +\infty)}^\diamond$ are such that $U f U^* = U g U^*$, then $f = g$.*

5. The class $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$

Proof. (a) This follows from Remark 4.6 and Example 4.2.

(b1) Let $G \in \mathcal{S}_{q,[\alpha,+\infty)}^\diamond[A]$. In view of Remark 4.6, there exists an $F \in \mathcal{S}_{q,[\alpha,+\infty)}^\diamond$ such that $G = A^\dagger A F A^\dagger A$. Let $f := U^* F U$. Because of Remark 3.6, then $f \in \mathcal{S}_{r,[\alpha,+\infty)}^\diamond$. In view of Remark B.5, we have $U U^* = A^\dagger A$. Thus, $G = U U^* F U U^* = U f U^*$. Hence, $\mathcal{S}_{q,[\alpha,+\infty)}^\diamond[A] \subseteq \{U f U^* \mid f \in \mathcal{S}_{r,[\alpha,+\infty)}^\diamond\}$.

Conversely, let $f \in \mathcal{S}_{r,[\alpha,+\infty)}^\diamond$. In view of Remark 3.6, then $U f U^* \in \mathcal{S}_{q,[\alpha,+\infty)}^\diamond$. Now we consider an arbitrary $x \in \mathcal{N}(A)$. In view of the construction of U and the relation $[\mathcal{N}(A)]^\perp = \mathcal{R}(A^*)$, we get $U^* x = O_{r \times 1}$. Thus, $x \in \mathcal{N}(U^*)$. Consequently, for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$ we get $\mathcal{N}(A) \subseteq \mathcal{N}((U f U^*)(z))$. The application of Lemma 4.3 yields now $\{U f U^* \mid f \in \mathcal{S}_{r,[\alpha,+\infty)}^\diamond\} \subseteq \mathcal{S}_{q,[\alpha,+\infty)}^\diamond[A]$. This completes the proof of (b1).

(b2) In view of Remark B.5, we have $U^* U = I_r$. Thus $U f U^* = U g U^*$ implies $f = g$. \square

5. The class $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$

In this section, we consider particular subclasses of the class $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}$, which was introduced in (3.1) for $\kappa = 0$ and in (3.5) for $\kappa \in \mathbb{N} \cup \{\infty\}$. In view of Theorem 3.2, for each function F belonging to one of the classes $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}$ with some $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we can consider the $[\alpha, +\infty)$ -Stieltjes measure σ_F of F . Now taking Remark 1.1 into account, we turn our attention to subclasses of functions $F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)}$ with prescribed first $\kappa + 1$ power moments of the $[\alpha, +\infty)$ -Stieltjes measure σ_F .

For all $\alpha \in \mathbb{R}$, all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and each sequence $(s_j)_{j=0}^\kappa$ of complex $q \times q$ matrices, now we consider the class

$$\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa] := \left\{ F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)} \mid \sigma_F \in \mathcal{M}_{q,\kappa}^{\geq}[\alpha, +\infty); (s_j)_{j=0}^\kappa, = \right\}. \quad (5.1)$$

Remark 5.1. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. If $\iota \in \mathbb{N}_0 \cup \{\infty\}$ with $\iota \leq \kappa$ then it is readily checked that

$$\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa] \subseteq \mathcal{S}_{\iota,q,[\alpha,+\infty)}[(s_j)_{j=0}^\iota].$$

and

$$\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa] = \bigcap_{m=0}^{\kappa} \mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m].$$

Remark 5.2. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Let $F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$ with $[\alpha, +\infty)$ -Stieltjes measure σ_F . Then

$$s_0 = s_0^{(\sigma_F)} = \int_{[\alpha,+\infty)} t^0 \sigma_F(dt) = \sigma_F([\alpha, +\infty)).$$

Now we characterize those sequences for which the sets defined in (5.1) are non-empty.

Theorem 5.3. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa] \neq \emptyset$ if and only if $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$.*

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Proof. Combine (5.1) and (3.4) with Theorems 1.2 and 3.2. \square

Now we state a useful characterization of the set of functions given in (5.1).

Theorem 5.4. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. In view of (3.2) then*

$$\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa] = \left\{ S_\sigma \mid \sigma \in \mathcal{M}_{q,\kappa}^{\geq}[\alpha, +\infty); (s_j)_{j=0}^\kappa, =] \right\}.$$

Proof. Combine (5.1) with Theorem 3.2. \square

Theorem 5.4 shows that $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$ coincides with the solution set of Problem $\mathbf{S}[[\alpha, +\infty); (s_j)_{j=0}^\kappa, =]$, which is via $[\alpha, +\infty)$ -Stieltjes transform equivalent to the original Problem $\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^\kappa, =]$. Thus, the investigation of the set $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$ is a central theme of our further considerations. The next result contains essential information on the functions belonging to this set.

Proposition 5.5. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, and let $F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$. Then:*

- (a) $\mathcal{R}(F(z)) = \mathcal{R}(s_0)$ and $\mathcal{N}(F(z)) = \mathcal{N}(s_0)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (b) $[F(z)][F(z)]^\dagger = s_0 s_0^\dagger$ and $[F(z)]^\dagger [F(z)] = s_0^\dagger s_0$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (c) The function F belongs to the class $\mathcal{S}_{q,[\alpha,+\infty)}$ with

$$\mathcal{R}(s_0) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty))), \quad \mathcal{N}(s_0) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty))). \quad (5.2)$$

Proof. (a) In view of the choice of F , we get

$$F \in \mathcal{S}_{0,q,[\alpha,+\infty)} \quad (5.3)$$

and $\sigma_F \in \mathcal{M}_{q,\kappa}^{\geq}[[\alpha, +\infty); (s_j)_{j=0}^\kappa, =]$. Thus, we have $\sigma_F([\alpha, +\infty)) = s_0$. Combining this with (5.3), we obtain from Lemma 3.9 all assertions of (a).

(b) The assertions of (b) follow from (a) by application of Remark B.3.

(c) From (5.3) and (3.1) we get $F \in \mathcal{S}_{q,[\alpha,+\infty)}$. Now the assertions of (c) follow by combination of Proposition 2.5 with (a). \square

The next result establishes a connection to the class $\mathcal{S}_{q,[\alpha,+\infty)}^\diamond[s_0]$ introduced in Section 4.

Lemma 5.6. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa] \subseteq \mathcal{S}_{q,[\alpha,+\infty)}^\diamond[s_0]$.*

Proof. Let $F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$. Then $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}$ and the $[\alpha, +\infty)$ -Stieltjes measure σ_F of F belongs to $\mathcal{M}_{q,\kappa}^{\geq}[[\alpha, +\infty); (s_j)_{j=0}^\kappa, =]$. From (3.6) we get then $F \in \mathcal{S}_{q,[\alpha,+\infty)}^\diamond$ and, in view of (1.1), we have $s_0 = s_0^{(\sigma_F)} = \sigma_F([\alpha, +\infty))$. Taking into account Lemma 3.9, we conclude then $\mathcal{N}(s_0) = \mathcal{N}(\sigma_F([\alpha, +\infty))) = \mathcal{N}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. From Remark 4.4 we obtain then $F \in \mathcal{S}_{q,[\alpha,+\infty)}^\diamond[s_0]$. \square

6. On Hamburger-Nevanlinna-type results for $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$

Remark 5.7. Let $\alpha \in \mathbb{R}$ and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be a matrix-valued function. Then:

- (a) If $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}$, then $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}[(s_j)_{j=0}^0]$ with $s_0 := \sigma_F([\alpha, +\infty))$.
- (b) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. If $F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$, then $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}$ and $\sigma_F([\alpha, +\infty)) = s_0$.

6. On Hamburger-Nevanlinna-type results for $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$

This section should be compared with [11, Section 6], where we investigated matricial generalizations of the classical Hamburger-Nevanlinna theorems for the class $\mathcal{R}_q(\Pi_+)$ of $q \times q$ matrix-valued functions which are holomorphic in Π_+ and have non-negative Hermitian imaginary values at each point. Using these results, we were able to reformulate the truncated matricial Hamburger moment problem in [11] into a problem of determining all functions from special subclasses of $\mathcal{R}_q(\Pi_+)$ which have prescribed asymptotic expansion in some sector of Π_+ .

The main goal of this section can be described as follows. Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then we are going to reformulate the moment problem $\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^m, =]$ into a problem of determining all functions from special subclasses of $\mathcal{S}_{q,[\alpha,+\infty)}$ which have a prescribed asymptotic expansion in some sector of Π_+ . This aim can be realized by application of a Stieltjes-type version of a matricial Hamburger-Nevanlinna theorem. Before formulating the result, we introduce some notation. For each $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, each $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and each sequence $(s_j)_{j=0}^\kappa$ of complex $q \times q$ matrices let $\mathcal{M}_{\geq}^q[\Omega; (s_j)_{j=0}^\kappa, =]$ be the set of all $\sigma \in \mathcal{M}_{\geq,\kappa}^q(\Omega)$ such that $s_j^{(\sigma)} = s_j$ is fulfilled for each $j \in \mathbb{Z}_{0,m}$. For all $r \in (0, +\infty)$ and each $\delta \in (0, \pi/2]$, let

$$\Sigma_{r,\delta} := \{z \in \mathbb{C} \mid |z| \geq r \text{ and } \delta \leq \arg z \leq \pi - \delta\}.$$

Taking (3.2) into account, now we can formulate the following Stieltjes-type version of the Hamburger-Nevanlinna theorem:

Theorem 6.1. *Let Ω be a non-empty closed subset of \mathbb{R} , let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices.*

- (a) *Let $\sigma \in \mathcal{M}_{q,m}^{\geq}[\Omega; (s_j)_{j=0}^m, =]$ and let S_σ be the Ω -Stieltjes transform of σ . For all $\ell \in \mathbb{Z}_{0,m}$ and all $\delta \in (0, \pi/2]$, then*

$$\lim_{r \rightarrow +\infty} \sup_{z \in \Sigma_{r,\delta}} \left\| z^{\ell+1} \left[S_\sigma(z) + \sum_{j=0}^{\ell} z^{-(j+1)} s_j \right] \right\|_{\mathbb{E}} = 0.$$

- (b) *Suppose that $\inf\{|\inf \Omega|, |\sup \Omega|\} < +\infty$ and that the matrices s_0, s_1, \dots, s_m are Hermitian. Let $\sigma \in \mathcal{M}_{\geq}^q(\Omega)$ be such that the Ω -Stieltjes transform S_σ of σ fulfills*

$$\lim_{y \rightarrow +\infty} \left\| (iy)^{m+1} \left[S_\sigma(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] \right\|_{\mathbb{E}} = 0. \quad (6.1)$$

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Then σ belongs to $\mathcal{M}_{q,m}^\geq[\Omega; (s_j)_{j=0}^m, =]$.

Theorem 6.1 should be compared with [11, Theorem 6.1]. A closer look shows that Theorem 6.1 simultaneously works for all $m \in \mathbb{N}_0$ whereas in [11, Theorem 6.1] it is assumed that m is even.

Part (b) of Theorem 6.1 will be often applied in the following for $\Omega = [\alpha, +\infty)$. It contains a sufficient condition which implies that a function $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}$ is the $[\alpha, +\infty)$ -Stieltjes transform of a solution σ of Problem M $[[\alpha, +\infty); (s_j)_{j=0}^m, =]$.

The proof of part (a) of Theorem 6.1 can be done analogously to the proof in the case that $\Omega = \mathbb{R}$ and $m = 2n$ with some $n \in \mathbb{N}_0$ hold (see, e. g. [20], [4, Lemma 2.1]). We omit the details. Now we are going to prove part (b) of Theorem 6.1. In order to do this, we need some auxiliary results:

Remark 6.2. Let $m \in \mathbb{N}_0$, let \mathcal{G} be a subset of \mathbb{C} such that $\{iy \mid y \in (0, +\infty)\} \subseteq \mathcal{G}$, let $F: \mathcal{G} \rightarrow \mathbb{C}^{p \times q}$ be a matrix-valued function, and let $(s_j)_{j=0}^m$ be a sequence of complex $p \times q$ matrices such that

$$\lim_{y \rightarrow +\infty} (iy)^{\ell+1} \left[F(iy) + \sum_{j=0}^{\ell} (iy)^{-(j+1)} s_j \right] = O_{p \times q} \quad (6.2)$$

holds for $\ell = m$. Then one can easily check by induction that (6.2) is valid for each $\ell \in \mathbb{Z}_{0,m}$.

Lemma 6.3. *Let Ω be a non-empty closed subset of \mathbb{R} such that $\inf\{|\inf \Omega|, |\sup \Omega|\} < +\infty$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^{m-1}$ be a sequence of complex $q \times q$ matrices, and let s_m be a Hermitian $q \times q$ matrix. Let $\sigma \in \mathcal{M}_{q,m-1}^\geq[\Omega; (s_j)_{j=0}^{m-1}, =]$ be such that the Ω -Stieltjes transform S_σ of σ fulfills*

$$\lim_{y \rightarrow +\infty} (iy)^{m+1} \left[S_\sigma(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] = O_{q \times q}. \quad (6.3)$$

Then σ belongs to $\mathcal{M}_{q,m}^\geq[\Omega; (s_j)_{j=0}^m, =]$.

Proof. Because of (6.3) we have

$$s_m = - \lim_{n \rightarrow \infty} (in)^{m+1} \left[S_\sigma(in) + \sum_{j=0}^{m-1} (in)^{-(j+1)} s_j \right]. \quad (6.4)$$

For each $n \in \mathbb{N}$, let $f_n: \Omega \rightarrow \mathbb{C}$ and $g_n: \Omega \rightarrow \mathbb{C}$ be defined by

$$f_n(t) := \frac{n^2 t^m}{t^2 + n^2} - i \frac{n t^{m+1}}{t^2 + n^2} \quad \text{and} \quad g_n(t) := \frac{n^2 t^m}{t^2 + n^2}.$$

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For each $n \in \mathbb{N}$ we get in view of $\sigma \in \mathcal{M}_{q,m-1}^{\geq}[\Omega; (s_j)_{j=0}^{m-1}, =]$ then

$$\begin{aligned} & - (in)^{m+1} \left[S_\sigma(in) + \sum_{j=0}^{m-1} (in)^{-(j+1)} s_j \right] \\ &= - (in)^{m+1} \int_{\Omega} \frac{1}{t - in} \sigma(dt) - \sum_{j=0}^{m-1} (in)^{m-j} \int_{\Omega} t^j \sigma(dt) \\ &= \int_{\Omega} \frac{1}{t - in} \left[- (in)^{m+1} - \sum_{j=0}^{m-1} (in)^{m-j} t^j (t - in) \right] \sigma(dt) \\ &= \int_{\Omega} \frac{-int^m}{t - in} \sigma(dt) = \int_{\Omega} \frac{-int^{m+1} - (in)^2 t^m}{t^2 + n^2} \sigma(dt) = \int_{\Omega} f_n d\sigma \end{aligned}$$

and in particular $f_n \in \mathcal{L}^1(\Omega, \mathfrak{B}_{\Omega}, \sigma; \mathbb{C})$. Hence, for each $n \in \mathbb{N}$, because of $g_n = \operatorname{Re} f_n$ and Remark A.4 we obtain

$$g_n \in \mathcal{L}^1(\Omega, \mathfrak{B}_{\Omega}, \sigma; \mathbb{C}) \quad (6.5)$$

and

$$\operatorname{Re} \left(- (in)^{m+1} \left[S_\sigma(in) + \sum_{j=0}^{m-1} (in)^{-(j+1)} s_j \right] \right) = \int_{\Omega} g_n d\sigma. \quad (6.6)$$

Using (6.6), (6.4) and $s_m^* = s_m$ we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\sigma &= \lim_{n \rightarrow \infty} \operatorname{Re} \left(- (in)^{m+1} \left[S_\sigma(in) + \sum_{j=0}^{m-1} (in)^{-(j+1)} s_j \right] \right) \\ &= \operatorname{Re} \left(- \lim_{n \rightarrow \infty} (in)^{m+1} \left[S_\sigma(in) + \sum_{j=0}^{m-1} (in)^{-(j+1)} s_j \right] \right) = \operatorname{Re} s_m = s_m. \end{aligned} \quad (6.7)$$

Now we are going to prove that the function $g: \Omega \rightarrow \mathbb{C}$ given by $g(t) := t^m$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_{\Omega}, \sigma; \mathbb{C})$ and that $\lim_{n \rightarrow \infty} \int_{\Omega} g_n d\sigma = \int_{\Omega} g d\sigma$. Obviously, for each $t \in \Omega$ we have

$$\lim_{n \rightarrow \infty} g_n(t) = t^m = g(t). \quad (6.8)$$

First we consider the case

$$\inf\{|\inf \Omega|, |\sup \Omega|\} = |\inf \Omega|. \quad (6.9)$$

Then $\inf \Omega \in \mathbb{R}$ and $\alpha := \min\{0, \inf \Omega\}$ is a non-positive real number. Obviously,

$$[\alpha, 0] \cup (0, +\infty) = [\alpha, +\infty), \quad [\alpha, 0] \cap (0, +\infty) = \emptyset \quad \text{and} \quad \Omega \subseteq [\alpha, +\infty).$$

Thus the sets $\mathcal{A} := \Omega \cap [\alpha, 0]$ and $\mathcal{B} := \Omega \cap (0, +\infty)$ are disjoint and fulfill $\mathcal{A} \cup \mathcal{B} = \Omega$. Consequently,

$$1_{\mathcal{A}} + 1_{\mathcal{B}} = 1_{\Omega} \quad (6.10)$$

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where, for each $\mathcal{M} \in \mathfrak{B}_\Omega$, the notation $1_{\mathcal{M}}$ denotes the indicator function of \mathcal{M} (defined on Ω). For each $t \in \Omega$, from (6.8) we conclude that

$$\lim_{n \rightarrow \infty} (1_{\mathcal{A}} g_n)(t) = (1_{\mathcal{A}} g)(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} (1_{\mathcal{B}} g_n)(t) = (1_{\mathcal{B}} g)(t) \quad (6.11)$$

hold. For each $n \in \mathbb{N}$ and each $t \in \Omega$ the estimate $|(1_{\mathcal{A}} g_n)(t)| \leq 1_{\mathcal{A}}(t) \cdot |t|^m \leq |\alpha|^m 1_{\mathcal{A}}(t)$ is true. Since $|\alpha|^m 1_{\mathcal{A}}$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$ application of Proposition A.5 yields then $1_{\mathcal{A}} g \in \mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$, $1_{\mathcal{A}} g_n \in \mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$ for all $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \int_{\Omega} 1_{\mathcal{A}} g_n d\sigma = \int_{\Omega} 1_{\mathcal{A}} g d\sigma. \quad (6.12)$$

We consider an arbitrary $x \in \mathbb{C}^q$. For each $n \in \mathbb{N}$, from (6.5) we see that $1_{\mathcal{B}} g_n$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, x^* \sigma x; \mathbb{R})$. Moreover, one can easily check that $0 \leq 1_{\mathcal{B}} g_n \leq 1_{\mathcal{B}} g_{n+1}$ holds for every $n \in \mathbb{N}$. Because of $1_{\mathcal{A}} + 1_{\mathcal{B}} = 1_\Omega$, we have $g_n - 1_{\mathcal{A}} g_n = 1_{\mathcal{B}} g_n$ for each $n \in \mathbb{N}$. Using this, (6.7), and (6.12), we conclude that

$$x^* \left(s_m - \int_{\Omega} 1_{\mathcal{A}} g d\sigma \right) x = \lim_{n \rightarrow \infty} \int_{\Omega} 1_{\mathcal{B}} g_n d(x^* \sigma x). \quad (6.13)$$

Hence, in view of (6.11), (6.13), and the monotone convergence theorem, we get $1_{\mathcal{B}} g \in \mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, x^* \sigma x; \mathbb{R})$ and $\lim_{n \rightarrow \infty} \int_{\Omega} 1_{\mathcal{B}} g_n d(x^* \sigma x) = \int_{\Omega} 1_{\mathcal{B}} g d(x^* \sigma x)$. Consequently, $1_{\mathcal{B}} g \in \mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{R})$ and

$$\lim_{n \rightarrow \infty} \int_{\Omega} 1_{\mathcal{B}} g_n d\sigma = \int_{\Omega} 1_{\mathcal{B}} g d\sigma \quad (6.14)$$

hold. Since (6.10) the identity $g = 1_{\mathcal{A}} g + 1_{\mathcal{B}} g$ is valid and since the functions $1_{\mathcal{A}} g$ and $1_{\mathcal{B}} g$ both belong to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{R})$ we get

$$g \in \mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{R}) \quad (6.15)$$

and

$$\int_{\Omega} g d\sigma = \int_{\Omega} 1_{\mathcal{A}} g d\sigma + \int_{\Omega} 1_{\mathcal{B}} g d\sigma.$$

From (6.12), (6.14), (6.10) and (6.7) we infer then

$$\begin{aligned} \int_{\Omega} t^m \sigma(dt) &= \int_{\Omega} g d\sigma = \lim_{n \rightarrow \infty} \int_{\Omega} 1_{\mathcal{A}} g_n d\sigma + \lim_{n \rightarrow \infty} \int_{\Omega} 1_{\mathcal{B}} g_n d\sigma \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} (1_{\mathcal{A}} g_n + 1_{\mathcal{B}} g_n) d\sigma = \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\sigma = s_m. \end{aligned} \quad (6.16)$$

Now we consider the case $\inf\{|\inf \Omega|, |\sup \Omega|\} = |\sup \Omega|$. Then we have $\sup \Omega \in \mathbb{R}$ and $\beta := \max\{0, \sup \Omega\}$ is a non-negative real number. Obviously, $[0, \beta] \cup (-\infty, 0) = (-\infty, \beta]$, $[0, \beta] \cap (-\infty, 0) = \emptyset$, and $\Omega \subseteq (-\infty, \beta]$. Thus the sets $\mathcal{C} := \Omega \cap [0, \beta]$ and $\mathcal{D} := \Omega \cap (-\infty, 0)$ are disjoint and fulfill $\mathcal{C} \cup \mathcal{D} = \Omega$. Analogous to the case (6.9) one can prove then that (6.15) and (6.16) hold. Thus (6.15) and (6.16) are proved for each case. Because of $\sigma \in \mathcal{M}_{q,m-1}^{\geq}[\Omega; (s_j)_{j=0}^{m-1}, =]$ and (6.16), we conclude then that σ belongs to $\mathcal{M}_{q,m}^{\geq}[\Omega; (s_j)_{j=0}^m, =]$. \square

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Proof of part (b) of Theorem 6.1. Using Lemma 3.1(c) one can easily check that

$$s_0^{(\sigma)} = \sigma(\Omega) = -i \lim_{y \rightarrow +\infty} y S_\sigma(iy) = - \left[\lim_{y \rightarrow +\infty} iy S_\sigma(iy) + s_0 \right] + s_0.$$

In view of (6.1), the application of Remark 6.2 yields then

$$s_0^{(\sigma)} = - \lim_{y \rightarrow +\infty} iy \left[S_\sigma(iy) + (iy)^{-1} s_0 \right] + s_0 = s_0.$$

Hence σ belongs to $\mathcal{M}_{q,0}^{\geq}[\Omega; (s_j)_{j=0}^0, =]$. Consequently, there is an $\ell \in \mathbb{Z}_{0,m}$ such that σ belongs to $\mathcal{M}_{q,\ell}^{\geq}[\Omega; (s_j)_{j=0}^\ell, =]$. We consider the case $\ell < m$. Then (6.1) and Remark 6.2 provide us

$$\lim_{y \rightarrow +\infty} (iy)^{\ell+2} \left[S_\sigma(iy) + \sum_{j=0}^{\ell+1} (iy)^{-(j+1)} s_j \right] = O_{q \times q}.$$

Application of Lemma 6.3 yields then $\sigma \in \mathcal{M}_{q,\ell+1}^{\geq}[\Omega; (s_j)_{j=0}^{\ell+1}, =]$. Consequently, $\sigma \in \mathcal{M}_{q,m}^{\geq}[\Omega; (s_j)_{j=0}^m, =]$ is inductively proved. \square

Theorem 6.1 was inspired by Benaych-Georges [1, Theorem 1.3]. This result was obtained in the context of free probability in the scalar case for the so-called R -transform. The R -transform realizes a bijective correspondence between the set of probability measures on \mathbb{R} and some set of holomorphic functions. The R -transform is different from our Ω -Stieltjes transform in (3.2). For this reason, even in the scalar case Theorem 6.1 is different from [1, Theorem 1.3]. However in a wider sense both results have some common features. In particular, the assumption of the semiboundedness of the support of the measure in part (b) of Theorem 6.1 was suggested by [1, Theorem 1.3(b)].

Now we give corollaries of the Stieltjes-type matricial Hamburger-Nevanlinna Theorem 6.1 for the particular case $\Omega = [\alpha, +\infty)$.

Corollary 6.4. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices, and let $F \in \mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$. For all $\ell \in \mathbb{Z}_{0,m}$ and all $\delta \in (0, \pi/2]$, then*

$$\lim_{r \rightarrow +\infty} \sup_{z \in \Sigma_{r,\delta}} \left\| z^{\ell+1} \left[F(z) + \sum_{j=0}^{\ell} z^{-(j+1)} s_j \right] \right\|_{\mathbb{E}} = 0.$$

Proof. Let σ be the $[\alpha, +\infty)$ -Stieltjes measure of F and let S_σ be the $[\alpha, +\infty)$ -Stieltjes transform of σ . Then $S_\sigma = F$ and, in view of (5.1), furthermore $\sigma \in \mathcal{M}_{q,m}^{\geq}[[\alpha, +\infty); (s_j)_{j=0}^m, =]$. The application of Theorem 6.1(a) completes the proof. \square

Corollary 6.5. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m$ be a sequence of Hermitian $q \times q$ matrices, and let $F \in \mathcal{S}_{q,[\alpha,+\infty)}$ be such that*

$$\lim_{y \rightarrow +\infty} (iy)^{m+1} \left[F(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] = O_{q \times q}. \quad (6.17)$$

Then F belongs to the class $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$.

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Proof. Remark 6.2 shows that (6.17) holds for $m = 0$. This implies that $\sup_{y \in [1, +\infty)} y \|F(iy)\|_E < +\infty$ is valid, i.e., F belongs to $\mathcal{S}_{0,q,[\alpha, +\infty)}$. In particular, $S_\sigma = F$, where σ is the $[\alpha, +\infty)$ -Stieltjes measure of F and S_σ is the $[\alpha, +\infty)$ -Stieltjes transform of σ . Using Theorem 6.1(b) and (5.1), the proof is finished. \square

Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of Hermitian $q \times q$ matrices. Then our preceding considerations lead us to the desired description of the set $\mathcal{S}_{m,q,[\alpha, +\infty)}[(s_j)_{j=0}^m]$.

Theorem 6.6. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of Hermitian $q \times q$ matrices. Then*

$$\begin{aligned} \mathcal{S}_{m,q,[\alpha, +\infty)}[(s_j)_{j=0}^m] \\ = \left\{ F \in \mathcal{S}_{q,[\alpha, +\infty)} \left| \lim_{y \rightarrow +\infty} (iy)^{m+1} \left[F(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] = O_{q \times q} \right. \right\}. \end{aligned}$$

Proof. Combine Corollaries 6.4 and 6.5. \square

Theorem 6.6 is the main result of this section. In combination with Theorem 5.4, we recognize now that Problem M $[[\alpha, +\infty); (s_j)_{j=0}^m, =]$ via $[\alpha, +\infty)$ -Stieltjes transform can be reformulated into an equivalent problem of prescribing the asymptotic behavior on the positive imaginary axis of functions belonging to $\mathcal{S}_{q,[\alpha, +\infty)}$, with the aid of the given sequence $(s_j)_{j=0}^m$ from $\mathbb{C}^{q \times q}$.

Theorem 6.6 should be compared with [11, Theorem 6.5]. A closer look shows that Theorem 6.6 works for all $m \in \mathbb{N}_0$, whereas in [11, Theorem 6.5] it is assumed that m is even. In [11], the case of an odd $m \in \mathbb{N}_0$ requires a separate treatment (see [11, Theorem 6.6, Proposition 6.7]).

7. On a Schur-type algorithm for sequences of complex $p \times q$ matrices

In this section, we recall some essential facts on the Schur-type algorithm for sequences from $\mathbb{C}^{p \times q}$, which was introduced and investigated in [12]. The elementary step of this algorithm is based on the use of the reciprocal sequence of a finite or infinite sequence from $\mathbb{C}^{p \times q}$. For this reason, we first remember the definition of the reciprocal sequence:

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then the sequence $(s_j^\#)_{j=0}^\kappa$ given by $s_0^\# := s_0^\dagger$ and, for all $k \in \mathbb{Z}_{1,\kappa}$, recursively by

$$s_k^\# := -s_0^\dagger \sum_{j=0}^{k-1} s_{k-j} s_j^\#,$$

is called the *reciprocal sequence corresponding to $(s_j)_{j=0}^\kappa$* . For a detailed treatment of the concept of reciprocal sequences, we refer the reader to [15]. Furthermore, we recall the notion of $[+, \alpha]$ -transformation for sequences of complex matrices (cf. [12, Definition 4.1]):

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Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Let $s_{-1} := O_{q \times q}$. Then the sequence $(s_j^{[+, \alpha]})_{j=0}^\kappa$ given by

$$s_j^{[+, \alpha]} := -\alpha s_{j-1} + s_j \quad (7.1)$$

is called the $[+, \alpha]$ -transform of $(s_j)_{j=0}^\kappa$. In order to prepare the basic construction, we use the reciprocal sequence corresponding to the $[+, \alpha]$ -transform of a sequence of complex matrices:

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with $[+, \alpha]$ -transform $(u_j)_{j=0}^\kappa$. Then the sequence $(s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ is given by

$$s_j^{[\sharp, \alpha]} := u_j^\sharp,$$

i. e., the sequence $(s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ is defined to be the reciprocal sequence corresponding to the $[+, \alpha]$ -transform of $(s_j)_{j=0}^\kappa$. Now we explain the elementary step of the Schur-type algorithm under consideration (cf. [12, Definition 7.1]):

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then the sequence $(t_j)_{j=0}^{\kappa-1}$ given by

$$t_j := -s_0 s_{j+1}^{[\sharp, \alpha]} s_0 \quad (7.2)$$

is called the α -Schur-transform (or short α -S-transform) of $(s_j)_{j=0}^\kappa$.

Remark 7.1 (cf. [12, Remark 7.3]). Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with α -S-transform $(t_j)_{j=0}^{\kappa-1}$. For all $m \in \mathbb{Z}_{1, \kappa}$, then the sequence $(t_j)_{j=0}^{m-1}$ depends only on the matrices s_0, s_1, \dots, s_m and, hence, it coincides with the α -S-transform of $(s_j)_{j=0}^m$.

The repeated application of the α -Schur-transform generates in a natural way a corresponding algorithm for (finite or infinite) sequences of complex $p \times q$ matrices (cf. [12, Definition 8.1]):

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^{[0, \alpha]})_{j=0}^\kappa$ given by $s_j^{[0, \alpha]} := s_j$ is called the 0-th α -S-transform of $(s_j)_{j=0}^\kappa$. In the case $\kappa \geq 1$, for all $k \in \mathbb{Z}_{1, \kappa}$, the k -th α -S-transform $(s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$ of $(s_j)_{j=0}^\kappa$ is recursively defined to be the α -Schur-transform of the $(k-1)$ -th α -S-transform $(s_j^{[k-1, \alpha]})_{j=0}^{\kappa-(k-1)}$ of $(s_j)_{j=0}^\kappa$.

One of the central properties of the just introduced Schur-type algorithm is that it preserves the α -Stieltjes non-negative definite extendability of sequences of matrices. This is the content of the following result.

Theorem 7.2 (cf. [12, Theorem 8.10]). *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq}$, and let $k \in \mathbb{Z}_{0, \kappa}$. Then the k -th α -S-transform $(s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$ of $(s_j)_{j=0}^\kappa$ belongs to $\mathcal{K}_{q, \kappa-k, \alpha}^{\geq, e}$.*

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In our considerations below, the special parametrization introduced in [9] will play an essential role, the so-called α -Stieltjes parametrization. For the convenience of the reader, we recall this notion. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For every choice of integers $\ell, m \in \mathbb{N}_0$ with $\ell \leq m \leq \kappa$, let

$$y_{\ell,m} := \begin{bmatrix} s_\ell \\ s_{\ell+1} \\ \vdots \\ s_m \end{bmatrix} \quad \text{and} \quad z_{\ell,m} := [s_\ell, s_{\ell+1}, \dots, s_m].$$

For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let

$$H_n := [s_{j+k}]_{j,k=0}^n$$

and, for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let

$$K_n := [s_{j+k+1}]_{j,k=0}^n.$$

Definition 7.3 (cf. [9, Definition 4.2]). Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then the sequence $(Q_j)_{j=0}^\kappa$ given by $Q_0 := s_0$ and

$$Q_{2k} := s_{2k} - z_{k,2k-1} H_{k-1}^\dagger y_{k,2k-1}$$

for all $k \in \mathbb{N}$ with $2k \leq \kappa$ and by $Q_1 := -\alpha s_0 + s_1$ and

$$Q_{2k+1} := (-\alpha s_{2k} + s_{2k+1}) - (-\alpha z_{k,2k-1} + z_{k+1,2k}) (-\alpha H_{k-1} + K_{k-1})^\dagger (-\alpha y_{k,2k-1} + y_{k+1,2k})$$

for all $k \in \mathbb{N}$ with $2k+1 \leq \kappa$ is called the α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$.

In [9], several important classes of sequences of complex $q \times q$ matrices were characterized in terms of their α -Stieltjes parametrization. From the view of this paper, the class $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ of α -Stieltjes non-negative definite extendable sequences is of extreme importance (see Theorems 1.2 and 5.3). In the case of a sequence $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, the α -Stieltjes parametrization can be generated by the above constructed Schur-type algorithm. This is the content of the following theorem.

Theorem 7.4 ([12, Theorem 9.15]). Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Then $(s_0^{[j,\alpha]})_{j=0}^\kappa$ is exactly the α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$.

An essential step in the further considerations of this paper can be described as follows. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ with α -Schur-transform $(t_j)_{j=0}^{\kappa-1}$. In view of Theorem 7.2, we get then $(t_j)_{j=0}^{\kappa-1} \in \mathcal{K}_{q,\kappa-1,\alpha}^{\geq,e}$. Thus, Theorem 5.3 yields that both sets $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$ and $\mathcal{S}_{\kappa-1,q,[\alpha,+\infty)}[(t_j)_{j=0}^{\kappa-1}]$ are non-empty. Then a central aspect of our strategy is based on the construction of a special bijective mapping $\mathcal{S}_{[+;\alpha,s_0]}$ with the property

$$\mathcal{S}_{[+;\alpha,s_0]} \left(\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa] \right) = \mathcal{S}_{\kappa-1,q,[\alpha,+\infty)}[(t_j)_{j=0}^{\kappa-1}].$$

8. Further considerations on the class of α -Stieltjes non-negative definite extendable sequences and some subclass

This mapping $\mathcal{S}_{[+;\alpha,s_0]}$ will help us to realize the basic step for our construction of a special Schur-type algorithm in the class $\mathcal{S}_{q,[\alpha,+\infty)}$, which stands in a bijective correspondence to the above described Schur-type algorithm for α -Stieltjes non-negative definite extendable sequences.

Against to this background we will use the inverse of the α -S-transform.

Definition 7.5 ([12, Definition 10.1]). Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. The sequence $(r_j)_{j=0}^{\kappa+1}$ with $r_0 := A$ recursively defined by

$$r_j := \alpha^j A + \sum_{\ell=1}^j \alpha^{j-\ell} A A^\dagger \left[\sum_{k=0}^{\ell-1} s_{\ell-k-1} A^\dagger r_k^{[+, \alpha]} \right]$$

is called the *inverse α -S-transform corresponding to $[(s_j)_{j=0}^\kappa, A]$* .

8. Further considerations on the class of α -Stieltjes non-negative definite extendable sequences and some subclass

In order to verify several interrelations between the algebraic and function theoretic versions of our Schur-type algorithm, we will need certain properties of α -Stieltjes non-negative definite extendable sequences of complex $q \times q$ matrices. Now we give a short summary of this material:

Lemma 8.1. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Then $s_j \in \mathbb{C}_H^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $s_{2k} \in \mathbb{C}_{\geq}^{q \times q}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$.

Proof. This is an immediate consequence of [9, Lemma 2.9]. □

Proposition 8.2 (cf. [9, Theorem 4.12(b)]). Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices with α -Stieltjes parametrization $(Q_j)_{j=0}^\kappa$. Then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$ if and only if $Q_j \in \mathbb{C}_{\geq}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $\mathcal{R}(Q_{j+1}) \subseteq \mathcal{R}(Q_j)$ for all $j \in \mathbb{Z}_{0,\kappa-2}$.

Now we recall a class of sequences of complex matrices which, as the considerations in [12] have shown, turned out to be extremely important in the framework of the above introduced Schur algorithm:

Definition 8.3 ([15, Definition 4.3]). Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. We then say that $(s_j)_{j=0}^\kappa$ is *dominated by its first term* (or, simply, that it is *first term dominant*) if $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(s_j)$ and $\bigcup_{j=0}^\kappa \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)$. The set of all first term dominant sequences $(s_j)_{j=0}^\kappa$ of complex $p \times q$ matrices will be denoted by $\mathcal{D}_{p \times q, \kappa}$.

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For a comprehensive investigation of first term dominant sequences, we refer the reader to the paper [15]. From the view of the Schur-type algorithm for sequences of matrices, the following result proved to be of central importance:

Proposition 8.4 ([12, Proposition 3.8(a)]). *Let $\alpha \in \mathbb{R}$ and let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e} \subseteq \mathcal{D}_{q \times q,\kappa}$.*

Now we turn our attention to further important subclasses of the class of all α -Stieltjes non-negative definite sequences. Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$. A sequence $(s_j)_{j=0}^m$ of complex $q \times q$ matrices is called α -Stieltjes positive definite if in the case $m = 0$ the block Hankel matrix H_0 is positive Hermitian, in the case $m = 2n$ with some $n \in \mathbb{N}$ the block Hankel matrices H_n and $-\alpha H_{n-1} + K_{n-1}$ are positive Hermitian, and in the case $m = 2n + 1$ with some $n \in \mathbb{N}_0$ the block Hankel matrices H_n and $-\alpha H_n + K_n$ are positive Hermitian, respectively. A sequence $(s_j)_{j=0}^\infty$ of complex $q \times q$ matrices is called α -Stieltjes positive definite if for all $m \in \mathbb{N}_0$ the sequence $(s_j)_{j=0}^m$ is α -Stieltjes positive definite. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we will write $\mathcal{K}_{q,\kappa,\alpha}^>$ for the set of all α -Stieltjes positive definite sequences $(s_j)_{j=0}^\kappa$ of complex $q \times q$ matrices.

Proposition 8.5 ([9, Proposition 2.20]). *Let $\alpha \in \mathbb{R}$ and let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathcal{K}_{q,\kappa,\alpha}^> \subseteq \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$.*

Proposition 8.6 ([9, Theorem 4.12(d)]). *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices with α -Stieltjes parametrization $(Q_j)_{j=0}^\kappa$. Then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^>$ if and only if $Q_j \in \mathbb{C}_{>}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$.*

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The main goal of this section is to prepare the elementary step of our Schur-type algorithm for the class $\mathcal{S}_{q,[\alpha,+\infty)}$. We will be led to a situation which, roughly speaking, looks as follows: Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{p \times q}$ and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{p \times q}$. Then the matrix-valued functions $F^{[+, \alpha, A]}: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{p \times q}$ and $F^{[-, \alpha, A]}: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{p \times q}$ which are defined by

$$F^{[+, \alpha, A]}(z) := -A \left(I_q + (z - \alpha)^{-1} [F(z)]^\dagger A \right) \quad (9.1)$$

and

$$F^{[-, \alpha, A]}(z) := -(z - \alpha)^{-1} A \left[I_q + A^\dagger F(z) \right]^\dagger, \quad (9.2)$$

respectively, will be central objects in our further considerations.

Against to the background of our later considerations, the matrix-valued functions $F^{[+, \alpha, A]}$ and $F^{[-, \alpha, A]}$ are called the (α, A) -Schur-Stieltjes transform of F and the inverse (α, A) -Schur-Stieltjes transform of F .

The generic case studied here concerns the situation where $p = q$, A is a complex $q \times q$ matrix with later specified properties and $F \in \mathcal{S}_{q,[\alpha,+\infty)}$.

The use of the transforms introduced in (9.2) was inspired by some considerations in the papers Chen/Hu [5, 17]. In particular, we mention [17, formula (2.3)]. Before treating

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more general aspects we state some relevant concrete examples for the constructions given by formulas (9.1) and (9.2). First we illustrate the transformations given in (9.1) and (9.2) by some examples.

Example 9.1. Let $A \in \mathbb{C}^{q \times q}$. In view of (9.1), then:

- (a) Let $\gamma \in \mathbb{C}^{q \times q}$ and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $F(z) := \gamma$. Then $F^{[+, \alpha, A]}(z) = -A + \frac{1}{\alpha - z} A \gamma^\dagger A$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.
- (b) Let $M \in \mathbb{C}^{q \times q}$, let $\tau \in [\alpha, +\infty)$, and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $F(z) := \frac{1+\tau-\alpha}{\tau-z} M$. Then

$$F^{[+, \alpha, A]}(z) = \left(\frac{1}{1+\tau-\alpha} A M^\dagger A - A \right) + \frac{1}{\alpha - z} \left(\frac{\tau - \alpha}{1+\tau-\alpha} A M^\dagger A \right)$$

for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

For all $\tau \in [\alpha, +\infty)$, let δ_τ be the Dirac measure defined on $\mathfrak{B}_{[\alpha, +\infty)}$ with unit mass at τ . Furthermore, let the measure $o_q: \mathfrak{B}_{[\alpha, +\infty)} \rightarrow \mathbb{C}_{\geq}^{q \times q}$ be defined by $o_q(B) := O_{q \times q}$.

Example 9.2. Let $A \in \mathbb{C}^{q \times q}$. In view of Theorem 2.2 and Example 9.1, then:

- (a) Let $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $F(z) := \gamma$. Then $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $(\gamma_F, \mu_F) = (\gamma, o_q)$. If $-A \in \mathbb{C}_{\geq}^{q \times q}$, then $F^{[+, \alpha, A]} \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $(\gamma_{F^{[+, \alpha, A]}}, \mu_{F^{[+, \alpha, A]}}) = (-A, \delta_\alpha A \gamma^\dagger A)$.
- (b) Let $M \in \mathbb{C}_{\geq}^{q \times q}$, let $\tau \in [\alpha, +\infty)$, and let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $F(z) := \frac{1+\tau-\alpha}{\tau-z} M$. Then $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $(\gamma_F, \mu_F) = (O_{q \times q}, \delta_\tau M)$. If $\frac{1}{1+\tau-\alpha} A M^\dagger A \geq A$, then $F^{[+, \alpha, A]} \in \mathcal{S}_{q, [\alpha, +\infty)}$ and

$$(\gamma_{F^{[+, \alpha, A]}}, \mu_{F^{[+, \alpha, A]}}) = \left(\frac{1}{1+\tau-\alpha} A M^\dagger A - A, \frac{\tau - \alpha}{1+\tau-\alpha} A M^\dagger A \right).$$

Example 9.3. Let $A, \gamma \in \mathbb{C}^{q \times q}$ and let $G: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $G(z) := \gamma$. In view of (9.2), then $G^{[-, \alpha, A]}(z) = \frac{1}{\alpha - z} A(I_q + A^\dagger \gamma)^\dagger$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

Example 9.4. Let $A \in \mathbb{C}^{q \times q}$, let $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ with $A(I_q + A^\dagger \gamma)^\dagger \in \mathbb{C}_{\geq}^{q \times q}$ and let $G: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $G(z) := \gamma$. In view of Theorem 2.2 and Example 9.3, then $G \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $(\gamma_G, \mu_G) = (\gamma, o_q)$, and, furthermore, $G^{[-, \alpha, A]} \in \mathcal{S}_{q, [\alpha, +\infty)}$ and

$$(\gamma_{G^{[-, \alpha, A]}}, \mu_{G^{[-, \alpha, A]}}) = \left(O_{q \times q}, O_{q \times q}, \delta_\alpha A(I_q + A^\dagger \gamma)^\dagger \right).$$

A central theme of this paper is to choose, for a given function $F \in \mathcal{S}_{q, [\alpha, +\infty)}$, special matrices $A \in \mathbb{C}^{q \times q}$ such that the function $F^{[+, \alpha, A]}$ and $F^{[-, \alpha, A]}$, respectively, belongs to $\mathcal{S}_{q, [\alpha, +\infty)}$. The following result provides a first contribution to this topic.

Proposition 9.5. *Let $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ and let $A \in \mathbb{C}^{q \times q}$ be such that $-A \in \mathbb{C}_{\geq}^{q \times q}$. Then $F^{[+, \alpha, A]} \in \mathcal{S}_{q, [\alpha, +\infty)}$.*

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Proof. Let $G: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$G(z) := -A. \quad (9.3)$$

In view of $-A \in \mathbb{C}_{\geq}^{q \times q}$ and (9.3), we see from Theorem 2.2(b) then

$$G \in \mathcal{S}_{q, [\alpha, +\infty)}. \quad (9.4)$$

Let $H: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$H(z) := -(z - \alpha)^{-1} [F(z)]^\dagger. \quad (9.5)$$

We see from Proposition 2.7 then

$$H \in \mathcal{S}_{q, [\alpha, +\infty)}. \quad (9.6)$$

Taking (9.4), (9.6) and $A^* = A$ into account, we infer from [13, Remark 3.11] then

$$G + AHA \in \mathcal{S}_{q, [\alpha, +\infty)}. \quad (9.7)$$

Because of (9.1), (9.3) and (9.5), we have

$$F^{[+, \alpha, A]} = G + AHA. \quad (9.8)$$

Using (9.8) and (9.7), we get $F^{[+, \alpha, A]} \in \mathcal{S}_{q, [\alpha, +\infty)}$. \square

Furthermore, we will show that under appropriate conditions the equations

$$(F^{[+, \alpha, A]})^{[-, \alpha, A]} = F \quad \text{and} \quad (G^{[-, \alpha, A]})^{[+, \alpha, A]} = G \quad (9.9)$$

hold true. The formulas in (9.9) show that the functions $F^{[+, \alpha, A]}$ and $G^{[-, \alpha, A]}$ form indeed a coupled pair of transformations. Furthermore it will be clear now the choice of our terminologies “ (α, A) -Schur-Stieltjes transform” and “inverse (α, A) -Schur-Stieltjes transform”. If all Moore-Penrose inverses in (9.1) and (9.2) would be indeed inverse matrices, then the equations in (9.9) could be confirmed by straightforward direct computations. Unfortunately, this is not the case in more general situations which are of interest for us. So we have to look for a convenient way to prove the equations in (9.9) for situations which will be relevant for us.

Now we verify that in important cases the formulas (9.1) and (9.2) can be rewritten as linear fractional transformations with appropriately chosen generating matrix-valued functions. The role of these generating functions will be played by the matrix polynomials $W_{\alpha, A}$ and $V_{\alpha, A}$ which are studied in Appendix D. In the sequel we use the terminology for linear fractional transformations of matrices which was introduced in Appendix C.

Lemma 9.6. *Let $\alpha \in \mathbb{R}$, let $F: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{p \times q}$ be a matrix-valued function, and let $A \in \mathbb{C}^{p \times q}$ be such that $\mathcal{R}(F(z)) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(F(z)) \subseteq \mathcal{N}(A)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Then $F(z) \in \mathcal{Q}_{[-(z-\alpha)A^\dagger, I_q - A^\dagger A]}$ and $F^{[+, \alpha, A]}(z) = \mathcal{S}_{W_{\alpha, A}(z)}^{(p, q)}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.*

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Proof. Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. The matrix $Y := (z - \alpha)F(z)$ fulfills $\mathcal{R}(Y) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(Y) \subseteq \mathcal{N}(A)$, which in view of Lemma B.4 implies $\mathcal{R}(Y) = \mathcal{R}(A)$ and $\mathcal{N}(Y) = \mathcal{N}(A)$. Hence, $YY^\dagger = AA^\dagger$ and $Y^\dagger Y = A^\dagger A$. Thus, we obtain

$$\begin{aligned} & (-Y^\dagger A + I_q - A^\dagger A)(-A^\dagger Y + I_q - A^\dagger A) \\ &= Y^\dagger AA^\dagger Y - Y^\dagger A(I_q - A^\dagger A) - (I_q - A^\dagger A)A^\dagger Y + (I_q - A^\dagger A)^2 \\ &= Y^\dagger YY^\dagger Y + I_q - A^\dagger A = Y^\dagger Y + I_q - Y^\dagger Y = I_q \end{aligned}$$

and

$$\begin{aligned} (Y + A)(-Y^\dagger A + I_q - A^\dagger A) &= -YY^\dagger A + Y - YA^\dagger A - AY^\dagger A + A(I_q - A^\dagger A) \\ &= -AA^\dagger A + Y - YY^\dagger Y - AY^\dagger A = -A(I_q + Y^\dagger A). \end{aligned}$$

In particular, $\det[-(z - \alpha)A^\dagger F(z) + I_q - A^\dagger A] \neq 0$ and $[-(z - \alpha)A^\dagger F(z) + I_q - A^\dagger A]^{-1} = -Y^\dagger A + I_q - A^\dagger A$. In view of (9.1), (D.1), and (C.1), the proof is complete. \square

The following application of Lemma 9.6 is important for our considerations in Section 12.

Proposition 9.7. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, and let $F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$. Then $F(z) \in \mathcal{Q}_{[-(z-\alpha)s_0^\dagger, I_q - s_0^\dagger s_0]}$ and $\mathcal{S}_{W_{\alpha,s_0}(z)}^{(q,q)}(F(z)) = F^{[+,\alpha,s_0]}(z)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.*

Proof. In view of Proposition 5.5(a) we have

$$\mathcal{R}(F(z)) = \mathcal{R}(s_0) \quad \text{and} \quad \mathcal{N}(F(z)) = \mathcal{N}(s_0). \quad (9.10)$$

Taking (9.10) into account, we infer from Lemma 9.6 the assertions. \square

Now we are going to consider the following situation which will turn out to be typical for larger parts of our future considerations. Let $A \in \mathbb{C}_{\geq}^{q \times q}$ and $G \in \mathcal{S}_{q,[\alpha,+\infty)}$ be such that

$$\mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(A). \quad (9.11)$$

Then our aim is to investigate the function $G^{[-,\alpha,A]}$ given by (9.2). We begin by rewriting formula (9.2) as linear fractional transformation. In the sequel we will often use the fact that for $G \in \mathcal{S}_{q,[\alpha,+\infty)}$ the matrix γ_G given via Theorem 2.2(a) is non-negative Hermitian.

Lemma 9.8. *Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}_{\geq}^{q \times q}$, and let $G \in \mathcal{S}_{q,[\alpha,+\infty)}$ be such that (9.11) holds. For all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then $G(z) \in \mathcal{Q}_{[(z-\alpha)A^\dagger, (z-\alpha)I_q]}$ and $G^{[-,\alpha,A]}(z) = \mathcal{S}_{V_{\alpha,A}(z)}^{(q,q)}(G(z))$.*

Proof. Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. We chose an arbitrary

$$v \in \mathcal{N}(I_q + A^\dagger G(z)). \quad (9.12)$$

According to Proposition 2.5 and (9.11), we obtain

$$\mathcal{R}(G(z)) \subseteq \mathcal{R}(A). \quad (9.13)$$

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Consequently, Remark B.3(b) implies

$$AA^\dagger G(z) = G(z). \quad (9.14)$$

Setting $F := G + A$, the application of (9.14) and (9.12) implies

$$[F(z)]v = A[I_q + A^\dagger G(z)]v = O_{q \times 1}. \quad (9.15)$$

Since the matrix γ_G is non-negative Hermitian, we have $\gamma_G + A \in \mathbb{C}_{\geq}^{q \times q}$. From Remark 2.3 we get then that F belongs to $\mathcal{S}_{q, [\alpha, +\infty)}$ and that $\gamma_F = \gamma_G + A$ and $\mu_F = \mu_G$ hold. Using Proposition 2.5, we conclude then that

$$\mathcal{N}(F(z)) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty))). \quad (9.16)$$

Furthermore, because of $\gamma_F \geq \gamma_G \geq O_{q \times q}$, we have $\mathcal{N}(\gamma_F) \subseteq \mathcal{N}(\gamma_G)$. From this, (9.16), and Proposition 2.5 we get then

$$\begin{aligned} \mathcal{N}(F(z)) &= \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty))) \\ &\subseteq \mathcal{N}(\gamma_G) \cap \mathcal{N}(\mu_G([\alpha, +\infty))) = \mathcal{N}(G(z)). \end{aligned} \quad (9.17)$$

From (9.15) and (9.17) we infer $v \in \mathcal{N}(G(z))$. Thus, we have

$$v = v + A^\dagger \cdot O_{q \times 1} = [I_q + A^\dagger G(z)]v = O_{q \times 1}.$$

Combining this with (9.12) we get $\mathcal{N}(I_q + A^\dagger G(z)) = \{O_{q \times 1}\}$. In view of (9.2), (D.2), and (C.1) this completes the proof. \square

The following result plays an essential role in our considerations in Section 12.

Proposition 9.9. *Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}_{\geq}^{q \times q}$ and let $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[A]$. Then $G(z) \in \mathcal{Q}_{[(z-\alpha)A^\dagger, (z-\alpha)I_q]}$ and $G^{[-, \alpha, A]}(z) = \mathcal{S}_{V_{\alpha, A}(z)}^{(q, q)}(G(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.*

Proof. First observe that $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$. In view of Remark 3.5, then $G \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $\gamma_G = O_{q \times q}$. From Remark 4.4 we obtain furthermore $\mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(A)$. Thus, we get $\mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(A)$. The application of Lemma 9.8 yields for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$ then

$$\det[(z - \alpha)A^\dagger G(z) + (z - \alpha)I_q] \neq 0$$

and furthermore $G^{[-, \alpha, A]}(z) = \mathcal{S}_{V_{\alpha, A}(z)}^{(q, q)}(G(z))$. \square

Now we formulate the first main result of this section. Assuming the situation of Lemma 9.8, we will obtain useful insights into the structure of the inverse (α, A) -Schur-Stieltjes transform of F .

Proposition 9.10. *Let $A \in \mathbb{C}_{\geq}^{q \times q}$, let $\alpha \in \mathbb{R}$, let $G \in \mathcal{S}_{q, [\alpha, +\infty)}$ be such that (9.11) holds, and let $u_0 := A(\gamma_G + A)^\dagger A$. Then $G^{[-, \alpha, A]}: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ given by (9.2) belongs to $\mathcal{S}_{0, q, [\alpha, +\infty)}[(u_j)_{j=0}^0]$ and fulfills $\mathcal{R}(G^{[-, \alpha, A]}(z)) = \mathcal{R}(A)$ and $\mathcal{N}(G^{[-, \alpha, A]}(z)) = \mathcal{N}(A)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.*

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Proof. Since the matrices γ_G and A are non-negative Hermitian, using Remark B.1(a) we conclude that u_0 is Hermitian. Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. According to Lemma 9.8, we get $\det[I_q + A^\dagger G(z)] \neq 0$ and taking (D.2) into account furthermore

$$G^{[-, \alpha, A]}(z) = -(z - \alpha)^{-1} A [I_q + A^\dagger G(z)]^{-1}. \quad (9.18)$$

By virtue of Proposition 2.5 and (9.11) we obtain (9.13). Consequently, Remark B.3(b) implies (9.14). From $\gamma_G \in \mathbb{C}_{\geq}^{q \times q}$ and $A \in \mathbb{C}_{\geq}^{q \times q}$ we get $\gamma_G + A \in \mathbb{C}_{\geq}^{q \times q}$. Thus, Remark 2.3 yields that $F := G + A$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}$ and that $\gamma_F = \gamma_G + A$ and

$$\mu_F = \mu_G \quad (9.19)$$

hold. Using Proposition 2.5, we conclude then (9.16). Moreover, because of $\gamma_F = \gamma_G + A \geq A \geq O_{q \times q}$, we get

$$\mathcal{R}(A) \subseteq \mathcal{R}(\gamma_F) \quad (9.20)$$

and $\mathcal{N}(\gamma_F) \subseteq \mathcal{N}(A)$. Because of (9.16), we obtain then

$$\mathcal{N}(F(z)) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty))) \subseteq \mathcal{N}(\gamma_F) \subseteq \mathcal{N}(A).$$

Consequently, Remark B.3(a) implies $AF^\dagger(z)F(z) = A$. Taking additionally into account (9.14), we get

$$AF^\dagger(z)A[I_q + A^\dagger G(z)] = AF^\dagger(z)[A + G(z)] = AF^\dagger(z)F(z) = A.$$

Hence, in view of (9.18), we conclude

$$\begin{aligned} G^{[-, \alpha, A]}(z) &= -(z - \alpha)^{-1} AF^\dagger(z)A[I_q + A^\dagger G(z)][I_q + A^\dagger G(z)]^{-1} \\ &= -(z - \alpha)^{-1} AF^\dagger(z)A. \end{aligned} \quad (9.21)$$

Let $H: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ be defined by $H(w) := -(w - \alpha)^{-1} F^\dagger(w)$. Because of (9.21), we have then

$$G^{[-, \alpha, A]}(z) = A[-(z - \alpha)^{-1} F^\dagger(z)]A = A^* H(z) A.$$

Since F belongs to $\mathcal{S}_{q, [\alpha, +\infty)}$, Proposition 2.7 yields $H \in \mathcal{S}_{q, [\alpha, +\infty)}$. From [13, Remark 3.11] we obtain then $A^* H A \in \mathcal{S}_{q, [\alpha, +\infty)}$, i. e., $G^{[-, \alpha, A]} \in \mathcal{S}_{q, [\alpha, +\infty)}$. Thanks to $F \in \mathcal{S}_{q, [\alpha, +\infty)}$, Proposition 2.4 provides us $\lim_{y \rightarrow +\infty} F(iy) = \gamma_F$, and Proposition 2.5 yields

$$\mathcal{R}(F(iy)) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty))) \quad \text{for each } y \in (0, +\infty). \quad (9.22)$$

In view of (9.19), (9.11) and (9.20), we get

$$\mathcal{R}(\mu_F([\alpha, +\infty))) = \mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(\gamma_F).$$

Consequently, using (9.22), we have

$$\mathcal{R}(F(iy)) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty))) = \mathcal{R}(\gamma_F)$$

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for each $y \in (0, +\infty)$. Hence, $\text{rank } F(iy) = \text{rank } \gamma_F$ hold for each $y \in (0, +\infty)$. Taking into account Lemma B.6, we obtain then $\lim_{y \rightarrow +\infty} F^\dagger(iy) = \gamma_F^\dagger$. Using this and (9.21), we get

$$\begin{aligned} O_{q \times q} &= -u_0 + u_0 = -A(\gamma_G + A)^\dagger A + u_0 = (-1) \cdot A\gamma_F^\dagger A + u_0 \\ &= \left(\lim_{y \rightarrow +\infty} iy \left[-(iy - \alpha)^{-1} \right] \right) A \left[\lim_{y \rightarrow +\infty} F^\dagger(iy) \right] A + u_0 \\ &= \lim_{y \rightarrow +\infty} \left(iy \left[-(iy - \alpha)^{-1} \right] A F^\dagger(iy) A + u_0 \right) \\ &= \lim_{y \rightarrow +\infty} iy \left[-(iy - \alpha)^{-1} A F^\dagger(iy) A + (iy)^{-1} u_0 \right] \\ &= \lim_{y \rightarrow +\infty} iy \left[G^{[-, \alpha, A]}(iy) + (iy)^{-1} u_0 \right]. \end{aligned}$$

By virtue of Corollary 6.5, then the function $G^{[-, \alpha, A]}$ belongs to $\mathcal{S}_{0, q, [\alpha, +\infty)}[(u_j)_{j=0}^0]$. Because of (9.18) we have $\mathcal{R}(G^{[-, \alpha, A]}(z)) = \mathcal{R}(A)$, whereas (9.21) yields $\mathcal{N}(A) \subseteq \mathcal{N}(G^{[-, \alpha, A]}(z))$. Hence, Lemma B.4 implies $\mathcal{N}(A) = \mathcal{N}(G^{[-, \alpha, A]}(z))$. \square

Now we indicate some generic situations in which the formulas in (9.9) are satisfied. We start with the first formula in (9.9).

Proposition 9.11. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ and let $A \in \mathbb{C}^{q \times q}$ be such that $\mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty))) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty))) \subseteq \mathcal{N}(A)$ hold. Then $(F^{[+, \alpha, A]})^{[-, \alpha, A]} = F$.*

Proof. Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. In view of Proposition 2.5 the matrix $X := F(z)$ fulfills $\mathcal{R}(X) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(X) \subseteq \mathcal{N}(A)$, which in view of Lemma B.4 implies $\mathcal{R}(X) = \mathcal{R}(A)$ and $\mathcal{N}(X) = \mathcal{N}(A)$. Hence, Remark B.3 yields $XX^\dagger = AA^\dagger$ and $X^\dagger X = A^\dagger A$. With $Y := F^{[+, \alpha, A]}(z)$ and (9.1) we obtain then

$$\begin{aligned} A^\dagger Y &= A^\dagger \left(-A \left[I_q + (z - \alpha)^{-1} X^\dagger A \right] \right) \\ &= -\frac{1}{z - \alpha} A^\dagger A X^\dagger A - A^\dagger A = -\frac{1}{z - \alpha} X^\dagger X X^\dagger A - A^\dagger A = -\frac{1}{z - \alpha} X^\dagger A - A^\dagger A. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &\left(-\frac{1}{z - \alpha} X^\dagger A + I_q - A^\dagger A \right) \left[-(z - \alpha) A^\dagger X + I_q - A^\dagger A \right] \\ &= X^\dagger A A^\dagger X - \frac{1}{z - \alpha} X^\dagger A (I_q - A^\dagger A) - (z - \alpha) (I_q - A^\dagger A) A^\dagger X + (I_q - A^\dagger A)^2 \\ &= X^\dagger X X^\dagger X + I_q - A^\dagger A = X^\dagger X + I_q - X^\dagger X = I_q. \end{aligned}$$

In particular, $\det[-(z - \alpha)^{-1} X^\dagger A + I_q - A^\dagger A] \neq 0$ and $[-(z - \alpha)^{-1} X^\dagger A + I_q - A^\dagger A]^{-1} =$

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$-(z - \alpha)A^\dagger X + I_q - A^\dagger A$. Finally, we get

$$\begin{aligned} -\frac{1}{z - \alpha}A(I_q + A^\dagger Y)^\dagger &= -\frac{1}{z - \alpha}A\left(I_q - \frac{1}{z - \alpha}X^\dagger A - A^\dagger A\right)^\dagger \\ &= -\frac{1}{z - \alpha}A\left(-\frac{1}{z - \alpha}X^\dagger A + I_q - A^\dagger A\right)^{-1} \\ &= -\frac{1}{z - \alpha}A\left[-(z - \alpha)A^\dagger X + I_q - A^\dagger A\right] \\ &= AA^\dagger X - \frac{1}{z - \alpha}A(I_q - A^\dagger A) = XX^\dagger X = X. \end{aligned}$$

In view of (9.1) and (9.2), the proof is complete. \square

Corollary 9.12. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, and let $F \in \mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$. Then $(F^{[+, \alpha, s_0]})^{[-, \alpha, s_0]} = F$.*

Proof. According to part (c) of Proposition 5.5, the function F belongs to $\mathcal{S}_{q,[\alpha,+\infty)}$ and (5.2) holds true. Thus, the application of Proposition 9.11 yields the assertion. \square

Now we turn our attention to the second formula in (9.9).

Proposition 9.13. *Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}_{\geq}^{q \times q}$, and let $G \in \mathcal{S}_{q,[\alpha,+\infty)}$ be such that (9.11) holds. Then $(G^{[-, \alpha, A]})^{[+, \alpha, A]} = G$.*

Proof. Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. According to (9.1), we have

$$(G^{[-, \alpha, A]})^{[+, \alpha, A]}(z) = -A\left(I_q + (z - \alpha)^{-1}\left[G^{[-, \alpha, A]}(z)\right]^\dagger A\right). \quad (9.23)$$

For each $w \in \mathbb{C} \setminus [\alpha, +\infty)$, from Lemma 9.8 we know that $\det[I_q + A^\dagger G(w)] \neq 0$ and that

$$G^{[-, \alpha, A]}(w) = -(w - \alpha)^{-1}A\left[I_q + A^\dagger G(w)\right]^{-1}. \quad (9.24)$$

From Proposition 9.10 we know that $G^{[-, \alpha, A]} \in \mathcal{S}_{0,q,[\alpha,+\infty)}$. Because of this and Corollary 3.3, we have $[G^{[-, \alpha, A]}(z)]^* = G^{[-, \alpha, A]}(\bar{z})$. From (9.24) we see that $\mathcal{R}(G^{[-, \alpha, A]}(\bar{z})) = \mathcal{R}(A)$ and, in view of $A^* = A$, consequently $\mathcal{N}(G^{[-, \alpha, A]}(z)) = [\mathcal{R}(A)]^\perp = \mathcal{N}(A)$. Hence, Remark B.3(b) shows that

$$A\left[G^{[-, \alpha, A]}(z)\right]^\dagger G^{[-, \alpha, A]}(z) = A. \quad (9.25)$$

Proposition 2.5 and (9.11) provide us (9.13) and, in view of Remark B.3(b), consequently (9.14). Using (9.23), (9.24), (9.25), and (9.14), we obtain

$$\begin{aligned} (G^{[-, \alpha, A]})^{[+, \alpha, A]}(z) &= -A + A\left[G^{[-, \alpha, A]}(z)\right]^\dagger\left[-(z - \alpha)^{-1}A\right] \\ &= -A + A\left[G^{[-, \alpha, A]}(z)\right]^\dagger\left(-(z - \alpha)^{-1}A\left[I_q + A^\dagger G(z)\right]^{-1}\right)\left[I_q + A^\dagger G(z)\right] \\ &= -A + A\left[G^{[-, \alpha, A]}(z)\right]^\dagger G^{[-, \alpha, A]}(z)\left[I_q + A^\dagger G(z)\right] \\ &= -A + A\left[I_q + A^\dagger G(z)\right] = AA^\dagger G(z) = G(z). \end{aligned} \quad \square$$

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Corollary 9.14. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, and let $G \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. Then $(G^{[-, \alpha, s_0]})^{[+, \alpha, s_0]} = G$.*

Proof. Lemma 8.1 yields $s_0 \in \mathbb{C}_{\geq}^{q \times q}$. Thus, taking Proposition 5.5(c) into account, the application of Proposition 9.13 yields the assertion. \square

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$$\mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$$

The central topic of this section can be described as follows. Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$. Then Theorem 5.3 tells us that the class $\mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$ is non-empty. If $F \in \mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$, then our interest is concentrated on the (α, s_0) -Schur-Stieltjes transform $F^{[+, \alpha, s_0]}$ of F . We will obtain a complete description of this object. In the case $m = 0$, we will show that $F^{[+, \alpha, s_0]}$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$ (see Theorem 10.1). Let us now consider the case $m \in \mathbb{N}$. If $(t_j)_{j=0}^{m-1}$ denotes the α -S-transform of $(s_j)_{j=0}^m$, then it will turn out (see Theorem 10.3 and Corollary 10.4) that $F^{[+, \alpha, s_0]}$ belongs to $\mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}]$. Our strategy to prove this is based on the application of Hamburger-Nevanlinna-type results for the class $\mathcal{S}_{q, [\alpha, +\infty)}$, which were developed in Section 6.

Now we start with the detailed treatment of the case $m = 0$.

Theorem 10.1. *Let $(s_j)_{j=0}^0 \in \mathcal{K}_{q, 0, \alpha}^{\geq, e}$ and let $F \in \mathcal{S}_{0, q, [\alpha, +\infty)}[(s_j)_{j=0}^0]$. Then $F^{[+, \alpha, s_0]}$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$.*

Proof. Denote by σ_F the $[\alpha, +\infty)$ -Stieltjes measure of F . From Remark 5.2 we infer

$$s_0 = \sigma_F([\alpha, +\infty)). \quad (10.1)$$

According to Proposition 3.4, the function $G: \mathbb{C} \setminus [\alpha, +\infty) \rightarrow \mathbb{C}^{q \times q}$ defined by $G(z) := -(z - \alpha)^{-1} F^\dagger(z)$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}$ and using (10.1) we get $\gamma_G = [\sigma_F([\alpha, +\infty))]^\dagger = s_0^\dagger$. In view of Lemma 8.1 we have $s_0 \in \mathbb{C}_{\geq}^{q \times q}$. Thus, $s_0^* = s_0$. From this and (9.1) we infer

$$F^{[+, \alpha, s_0]} = -s_0 + s_0^* G s_0. \quad (10.2)$$

Furthermore, $H := s_0^* G s_0$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}$ and $\gamma_H = s_0^* \gamma_G s_0$ by virtue of [13, Remark 3.11]. In view of $s_0^* = s_0$ then $\gamma_H = s_0$. Because of $\gamma_H + (-s_0) = O_{q \times q} \in \mathbb{C}_{\geq}^{q \times q}$ and Remark 2.3, then $F^{[+, \alpha, s_0]} \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $\gamma_{F^{[+, \alpha, s_0]}} = O_{q \times q}$. Hence, in view of Proposition 2.4, we obtain

$$\lim_{y \rightarrow +\infty} F^{[+, \alpha, s_0]}(iy) = \gamma_{F^{[+, \alpha, s_0]}} = O_{q \times q}.$$

Thus, (3.4) yields that $F^{[+, \alpha, s_0]}$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond$. From (10.2) we see moreover $F^{[+, \alpha, s_0]} s_0^\dagger s_0 = F^{[+, \alpha, s_0]}$. This finally shows $F^{[+, \alpha, s_0]} \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$. \square

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Corollary 10.2. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. Then $F^{[+, \alpha, s_0]} \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$.*

Proof. From Remark 5.1 we get $F \in \mathcal{S}_{0, q, [\alpha, +\infty)}[(s_j)_{j=0}^0]$, which, in view of Theorem 5.3, implies $(s_j)_{j=0}^0 \in \mathcal{K}_{q, 0, \alpha}^{\geq, e}$. Thus, the application of Theorem 10.1 completes the proof. \square

Now we turn our attention to the case $m \in \mathbb{N}$.

Theorem 10.3. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$ with α - S -transform $(t_j)_{j=0}^{m-1}$, and let $F \in \mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$. Then $F^{[+, \alpha, s_0]} \in \mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}]$.*

Proof. The basic strategy of our proof is to apply Corollary 6.5. From Theorem 7.2 we obtain $(t_j)_{j=0}^{m-1} \in \mathcal{K}_{q, m-1, \alpha}^{\geq, e}$. Lemma 8.1 yields then

$$t_j \in \mathbb{C}_H^{q \times q} \quad \text{for all } j \in \mathbb{Z}_{0, m-1}. \quad (10.3)$$

Furthermore, $F \in \mathcal{S}_{0, q, [\alpha, +\infty)}$ and the $[\alpha, +\infty)$ -Stieltjes measure σ_F of F belongs to $\mathcal{M}_{q, m}^{\geq}[[\alpha, +\infty); (s_j)_{j=0}^m, =]$. Remark 5.2 implies

$$s_0 = \sigma_F([\alpha, +\infty)). \quad (10.4)$$

By virtue of Corollary 10.2, we have $F^{[+, \alpha, s_0]} \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$ and hence

$$F^{[+, \alpha, s_0]} \in \mathcal{S}_{q, [\alpha, +\infty)}. \quad (10.5)$$

In view of $F \in \mathcal{S}_{0, q, [\alpha, +\infty)}$, the application of Propositions 3.4 and 2.4 and (10.4) yields

$$\lim_{y \rightarrow +\infty} (iy - \alpha)^{-1} F^\dagger(iy) = -s_0^\dagger. \quad (10.6)$$

From Corollary 6.4 we conclude that

$$\lim_{y \rightarrow +\infty} (iy)^{m+1} \left[F(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] = O_{q \times q}$$

and, consequently, in view of $\lim_{y \rightarrow +\infty} (iy)^{-1} = 0$, then

$$\begin{aligned} O_{q \times q} &= \left(1 - \alpha \left[\lim_{y \rightarrow +\infty} (iy)^{-1} \right] \right) \left(\lim_{y \rightarrow +\infty} (iy)^{m+1} \left[F(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] \right) \\ &= \lim_{y \rightarrow +\infty} (iy)^m (iy - \alpha) \left[F(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right]. \end{aligned} \quad (10.7)$$

Because of Lemma 3.9 and (10.4), for each $y \in (0, +\infty)$, we have

$$\mathcal{N}(F(iy)) \subseteq \mathcal{N}(\sigma_F([\alpha, +\infty))) = \mathcal{N}(s_0)$$

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and hence, in view of Remark B.3(b), then

$$s_0 F^\dagger(iy) F(iy) = s_0. \quad (10.8)$$

Using (9.1) and (10.8), for each $y \in (0, +\infty)$, we get

$$\begin{aligned} F^{[+, \alpha, s_0]}(iy) &= -s_0 \left[I_q + (iy - \alpha)^{-1} F^\dagger(z) s_0 \right] \\ &= -s_0 F^\dagger(iy) F(iy) - (iy - \alpha)^{-1} s_0 F^\dagger(iy) s_0 \\ &= (iy - \alpha)^{-1} s_0 F^\dagger(iy) [-s_0 - (iy - \alpha) F(iy)]. \end{aligned} \quad (10.9)$$

For each $y \in (0, +\infty)$ and each $j \in \mathbb{Z}_{0, m-1}$, from (10.8) and (7.2) we get

$$s_0 F^\dagger(iy) F(iy) s_0^\dagger t_j = s_0 s_0^\dagger t_j = t_j.$$

Consequently, for each $y \in (0, +\infty)$, we have

$$s_0 F^\dagger(iy) F(iy) s_0^\dagger \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k = \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k. \quad (10.10)$$

Taking into account (7.1), we see that

$$\begin{aligned} -(iy - \alpha) \sum_{j=0}^m (iy)^{-(j+1)} s_j &= - \left[(iy)^{-0} s_0 + \sum_{k=1}^m (iy)^{-k} (-\alpha s_{k-1} + s_k) - (iy)^{-(m+1)} \alpha s_m \right] \\ &= - \sum_{j=0}^m (iy)^{-j} s_j^{[+, \alpha]} + (iy)^{-(m+1)} \alpha s_m \end{aligned} \quad (10.11)$$

and

$$\sum_{j=0}^m (iy)^{-j} s_j^{[+, \alpha]} - s_0 = \sum_{j=0}^{m-1} (iy)^{-(j+1)} s_{j+1}^{[+, \alpha]}$$

hold true for each $y \in (0, +\infty)$. Furthermore, for each $y \in (0, +\infty)$, we obtain

$$\begin{aligned} - \sum_{j=0}^m \sum_{k=0}^{m-1} (iy)^{-j-(k+1)} s_j^{[+, \alpha]} s_0^\dagger t_k &= - \sum_{r=0}^{2m-1} \left[\sum_{\substack{k=0 \\ 0 \leq r-k \leq m}}^{m-1} (iy)^{-(r+1)} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \right] \\ &= - \sum_{r=0}^{2m-1} \left[\sum_{\substack{k=0 \\ r \geq k \geq r-m}}^{m-1} (iy)^{-(r+1)} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \right] = - \sum_{r=0}^{2m-1} \sum_{k=\max\{0, r-m\}}^{\min\{r, m-1\}} (iy)^{-(r+1)} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \\ &= - \sum_{r=0}^{m-1} \sum_{k=0}^r (iy)^{-(r+1)} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k - \sum_{r=m}^{2m-1} \sum_{k=r-m}^{m-1} (iy)^{-(r+1)} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k. \end{aligned} \quad (10.12)$$

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From Proposition 8.4 we obtain $(s_j)_{j=0}^m \in \mathcal{D}_{q \times q, m}$. Then [12, Lemma 7.11] yields

$$\sum_{k=0}^j s_{j-k}^{[+, \alpha]} s_0^\dagger t_k = s_{j+1}^{[+, \alpha]} \quad (10.13)$$

for each $j \in \mathbb{Z}_{0, m-1}$. Combining (10.12) and (10.13), we get

$$-\sum_{j=0}^m \sum_{k=0}^{m-1} (iy)^{-j-(k+1)} s_j^{[+, \alpha]} = -\sum_{j=0}^{m-1} (iy)^{-(j+1)} s_{j+1}^{[+, \alpha]} - \sum_{j=m}^{2m-1} \sum_{k=j-m}^{m-1} (iy)^{-(j+1)} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k.$$

For each $y \in (0, +\infty)$, from (10.9) and (10.10) we get

$$\begin{aligned} & (iy)^m \left[F^{[+, \alpha, s_0]}(iy) + \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k \right] \\ &= (iy)^m \left\{ (iy - \alpha)^{-1} s_0 F^\dagger(iy) [-s_0 - (iy - \alpha)F(iy)] \right. \\ & \quad \left. + (iy - \alpha)^{-1} (iy - \alpha) s_0 F^\dagger(iy) F(iy) s_0^\dagger \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k \right\} \quad (10.14) \\ &= (iy)^m (iy - \alpha)^{-1} s_0 F^\dagger(iy) \\ & \quad \times \left[-s_0 - (iy - \alpha)F(iy) + (iy - \alpha)F(iy) s_0^\dagger \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k \right]. \end{aligned}$$

Now we are going to derive an identity for the matrix in brackets on the right side of (10.14). For all $z \in \Pi_+$, let

$$M(z) := I_q - s_0^\dagger \sum_{k=0}^{m-1} z^{-(k+1)} t_k. \quad (10.15)$$

From (7.1) it follows $s_0^{[+, \alpha]} = s_0$. Combining this with (10.15), we obtain

$$\begin{aligned} & \left(\sum_{j=0}^m (iy)^{-j} s_j^{[+, \alpha]} \right) [M(iy)] = \left(\sum_{j=0}^m (iy)^{-j} s_j^{[+, \alpha]} \right) \left[I_q - s_0^\dagger \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k \right] \\ &= s_0^{[+, \alpha]} + \sum_{j=1}^m (iy)^{-j} s_j^{[+, \alpha]} - \sum_{j=0}^m \sum_{k=0}^{m-1} (iy)^{-j-(k+1)} s_j^{[+, \alpha]} s_0^\dagger t_k \\ &= s_0 + \sum_{j=0}^{m-1} (iy)^{-(j+1)} s_{j+1}^{[+, \alpha]} - \sum_{j=0}^{m-1} (iy)^{-(j+1)} s_{j+1}^{[+, \alpha]} - \sum_{j=m}^{2m-1} \sum_{k=j-m}^{m-1} (iy)^{-(j+1)} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \\ &= s_0 - \sum_{j=m}^{2m-1} \sum_{k=j-m}^{m-1} (iy)^{-(j+1)} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k. \end{aligned} \quad (10.16)$$

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Using (10.15), (10.11), (10.16), and (10.13), we infer then

$$\begin{aligned}
& -s_0 - (\mathrm{i}y - \alpha)F(\mathrm{i}y) + (\mathrm{i}y - \alpha)F(\mathrm{i}y)s_0^\dagger \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k = s_0 - (\mathrm{i}y - \alpha)F(\mathrm{i}y)M(\mathrm{i}y) \\
& = -s_0 - \left((\mathrm{i}y - \alpha) \left[F(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] - (\mathrm{i}y - \alpha) \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right) M(\mathrm{i}y) \\
& = -s_0 \\
& \quad - \left((\mathrm{i}y - \alpha) \left[F(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] - \sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} + (\mathrm{i}y)^{-(m+1)} \alpha s_m \right) M(\mathrm{i}y) \\
& = -s_0 - \left((\mathrm{i}y - \alpha) \left[F(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] + (\mathrm{i}y)^{-(m+1)} \alpha s_m \right) M(\mathrm{i}y) \\
& \quad - \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} \right] M(\mathrm{i}y) \\
& = -s_0 - \left((\mathrm{i}y - \alpha) \left[F(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] + (\mathrm{i}y)^{-(m+1)} \alpha s_m \right) M(\mathrm{i}y) \\
& \quad + s_0 - \sum_{j=m}^{2m-1} \sum_{k=j-m}^{m-1} (\mathrm{i}y)^{-(j+1)} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \\
& = - \sum_{j=m}^{2m-1} \sum_{k=j-m}^{m-1} (\mathrm{i}y)^{-(j+1)} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \\
& \quad - \left((\mathrm{i}y - \alpha) \left[F(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] + (\mathrm{i}y)^{-(m+1)} \alpha s_m \right) M(\mathrm{i}y)
\end{aligned} \tag{10.17}$$

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for all $y \in (0, +\infty)$. Because of (10.14), (10.17), and (10.15), we have

$$\begin{aligned}
& (\mathrm{i}y)^m \left[F^{[+, \alpha, s_0]}(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \\
&= (\mathrm{i}y)^m (\mathrm{i}y - \alpha)^{-1} s_0 F^\dagger(\mathrm{i}y) \left\{ - \sum_{j=m}^{2m-1} \sum_{k=j-m}^{m-1} (\mathrm{i}y)^{-(j+1)} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \right. \\
&\quad \left. - \left((\mathrm{i}y - \alpha) \left[F(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] + (\mathrm{i}y)^{-(m+1)} \alpha s_m \right) M(\mathrm{i}y) \right\} \\
&= s_0 (\mathrm{i}y - \alpha)^{-1} F^\dagger(\mathrm{i}y) \left\{ - \sum_{j=m}^{2m-1} \left[(\mathrm{i}y)^{m-j-1} \sum_{k=j-m}^{m-1} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \right] \right. \\
&\quad \left. - \left((\mathrm{i}y)^m (\mathrm{i}y - \alpha) \left[F(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] + (\mathrm{i}y)^{-1} \alpha s_m \right) \right. \\
&\quad \left. \times \left[I_q - s_0^\dagger \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right\} \tag{10.18}
\end{aligned}$$

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for all $y \in (0, +\infty)$. From (10.6), (10.7), and (10.18) we conclude

$$\begin{aligned}
O_{q \times q} &= s_0(-s_0^\dagger) \left\{ - \sum_{j=m}^{2m-1} \left(0 \cdot \sum_{k=j-m}^{m-1} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \right) \right. \\
&\quad \left. - (O_{q \times q} + 0 \cdot \alpha s_m) \left(I_q - s_0^\dagger \sum_{k=0}^{m-1} 0 \cdot t_k \right) \right\} \\
&= s_0 \left(\lim_{y \rightarrow +\infty} \left[(iy - \alpha)^{-1} F^\dagger(iy) \right] \right) \\
&\quad \times \left\{ - \sum_{j=m}^{2m-1} \left(\left[\lim_{y \rightarrow +\infty} (iy)^{m-j-1} \right] \sum_{k=j-m}^{m-1} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \right) \right. \\
&\quad \left. - \left[\lim_{y \rightarrow +\infty} \left((iy)^m (iy - \alpha) \left[F(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] \right) + \left[\lim_{y \rightarrow +\infty} (iy)^{-1} \right] \alpha s_m \right] \right. \\
&\quad \left. \times \left(I_q - s_0^\dagger \sum_{k=0}^{m-1} \left[\lim_{y \rightarrow +\infty} (iy)^{-(k+1)} \right] t_k \right) \right\} \\
&= \lim_{y \rightarrow +\infty} \left\{ s_0(iy - \alpha)^{-1} F^\dagger(iy) \left\{ - \sum_{j=m}^{2m-1} \left[(iy)^{m-j-1} \sum_{k=j-m}^{m-1} s_{j-k}^{[+, \alpha]} s_0^\dagger t_k \right] \right. \right. \\
&\quad \left. \left. - \left((iy)^m (iy - \alpha) \left[F(iy) + \sum_{j=0}^m (iy)^{-(j+1)} s_j \right] + (iy)^{-1} \alpha s_m \right) \right. \right. \\
&\quad \left. \left. \times \left[I_q - s_0^\dagger \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k \right] \right\} \right\} \\
&= \lim_{y \rightarrow +\infty} (iy)^m \left[F^{[+, \alpha, s_0]}(iy) + \sum_{k=0}^{m-1} (iy)^{-(k+1)} t_k \right].
\end{aligned} \tag{10.19}$$

In view of (10.3), (10.5), and (10.19), then Corollary 6.5 yields that $F^{[+, \alpha, s_0]}$ belongs to the class $\mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}]$. \square

Corollary 10.4. *Let $\alpha \in \mathbb{R}$, let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q, \infty, \alpha}^{\geq, e}$ with α -S-transform $(t_j)_{j=0}^\infty$, and let $F \in \mathcal{S}_{\infty, q, [\alpha, +\infty)}[(s_j)_{j=0}^\infty]$. Then $F^{[+, \alpha, s_0]}$ belongs to $\mathcal{S}_{\infty, q, [\alpha, +\infty)}[(t_j)_{j=0}^\infty]$.*

Proof. Combine Remarks 5.1 and 7.1 and Theorem 10.3. \square

Proposition 10.5. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ with α -S-transform $(t_j)_{j=0}^{\kappa-1}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. Then $F^{[+, \alpha, s_0]}$ belongs to $\mathcal{S}_{\kappa-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{\kappa-1}]$.*

Proof. Because of $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, we have $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$ for all $m \in \mathbb{Z}_{0, \kappa}$. Remark 5.1 yields $F \in \bigcap_{m=0}^\kappa \mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$. Thus, Theorem 10.3 shows then that $F^{[+, \alpha, s_0]}$

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belongs to $\bigcap_{m=0}^{\kappa-1} \mathcal{S}_{m, q, [\alpha, +\infty)}[(t_j)_{j=0}^m]$. Consequently, from Remark 5.1 then $F^{[+, \alpha, s_0]} \in \mathcal{S}_{\kappa-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{\kappa-1}]$ follows. \square

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Against to the background of Propositions 9.11 and 9.13, the considerations of Section 10 lead us to the study of two inverse problems which will be formulated now. Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$, and let $F \in \mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$. In the case $m = 0$, it was shown in Theorem 10.1 that $F^{[+, \alpha, s_0]}$ belongs to $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$. Now we start with a function $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$ and will show in Proposition 11.1 that $G^{[-, \alpha, s_0]} \in \mathcal{S}_{0, q, [\alpha, +\infty)}[(s_j)_{j=0}^0]$. Finally, we investigate the case $m \in \mathbb{N}$. Let $(t_j)_{j=0}^{m-1}$ be the α -S-transform of $(s_j)_{j=0}^{\kappa}$, then we know from Theorem 10.3 that $F^{[+, \alpha, s_0]} \in \mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}]$. So we are going to verify now that, for a function $G \in \mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}]$, its inverse (α, s_0) -Schur-Stieltjes transform $G^{[-, \alpha, s_0]}$ belongs to $\mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$ (see Theorem 11.3).

First we turn our attention to a detailed treatment of the case $m = 0$. An application of Proposition 9.10 will provide us quickly the desired result.

Proposition 11.1. *Let $\alpha \in \mathbb{R}$, let $(s_j)_{j=0}^0 \in \mathcal{K}_{q, 0, \alpha}^{\geq, e}$, and let $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$. Then $G^{[-, \alpha, s_0]} \in \mathcal{S}_{0, q, [\alpha, +\infty)}[(s_j)_{j=0}^0]$.*

Proof. Because of $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$, we have $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond$. In view of Remark 3.5, then $G \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $\gamma_G = O_{q \times q}$. From Remark 4.4 we obtain furthermore $\mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(s_0)$. Thus, we get $\mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(s_0)$ and $s_0 = s_0(\gamma_G + s_0)^\dagger s_0$. The application of Proposition 9.10 with $A := s_0$ provides us then $G^{[-, \alpha, s_0]} \in \mathcal{S}_{0, q, [\alpha, +\infty)}[(s_j)_{j=0}^0]$. \square

Corollary 11.2. *Let $(s_j)_{j=0}^0 \in \mathcal{K}_{q, 0, \alpha}^{\geq, e}$. For each $F \in \mathcal{S}_{0, q, [\alpha, +\infty)}[(s_j)_{j=0}^0]$, let*

$$\mathcal{S}_{[+; \alpha, s_0]}(F) := F^{[+, \alpha, s_0]}.$$

Then $\mathcal{S}_{[+; \alpha, s_0]}$ generates a bijective correspondence between $\mathcal{S}_{0, q, [\alpha, +\infty)}[(s_j)_{j=0}^0]$ and $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$. The inverse mapping $\mathcal{S}_{[+; \alpha, s_0]}^{-1}$ is given for each $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$ by $\mathcal{S}_{[+; \alpha, s_0]}^{-1}(G) := G^{[-, \alpha, s_0]}$.

Proof. Taking Corollary 9.12 and Corollary 9.14 into account, applying Theorem 10.1 and Proposition 11.1 completes the proof. \square

Now we turn our attention to the case $m \in \mathbb{N}$. Similar as in the proof of Theorem 10.3, we will use Hamburger-Nevanlinna-type results from Section 6.

Theorem 11.3. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ with α -S-transform $(t_j)_{j=0}^{m-1}$, and let $G \in \mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}]$. Then $G^{[-, \alpha, s_0]}$ belongs to $\mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$.*

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Proof. The basic strategy of our proof is to apply Corollary 6.5. Lemma 8.1 yields

$$s_0 \in \mathbb{C}_{\geq}^{q \times q}. \quad (11.1)$$

From Theorem 7.2 we get $(t_j)_{j=0}^{m-1} \in \mathcal{K}_{q, m-1, \alpha}^{\geq, e}$. According to Proposition 5.5(c) and (7.2) we have then $G \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $\mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G([\alpha, +\infty))) = \mathcal{R}(t_0)$, whereas (7.2) yields $\mathcal{R}(t_0) \subseteq \mathcal{R}(s_0)$. Thus

$$\mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G([\alpha, +\infty))) \subseteq \mathcal{R}(s_0). \quad (11.2)$$

Consequently, in view of (11.1) and (11.2), we infer from Proposition 9.10, that $G^{[-, \alpha, s_0]} \in \mathcal{S}_{0, q, [\alpha, +\infty)}[(u_j)_{j=0}^0]$, where $u_0 := s_0(\gamma_G + s_0)^\dagger s_0$. In particular,

$$G^{[-, \alpha, s_0]} \in \mathcal{S}_{q, [\alpha, +\infty)}. \quad (11.3)$$

The application of Corollary 6.4 provides us

$$\lim_{y \rightarrow +\infty} (\mathrm{i}y)^\ell \left[G(\mathrm{i}y) + \sum_{k=0}^{\ell-1} (\mathrm{i}y)^{-(k+1)} t_k \right] = O_{q \times q} \quad \text{for each } \ell \in \mathbb{Z}_{1, m}. \quad (11.4)$$

In particular, using (11.4) for $\ell = 1$, we get

$$\begin{aligned} I_q &= I_q + s_0^\dagger (0 \cdot O_{q \times q} - 0 \cdot t_0) \\ &= I_q + s_0^\dagger \left\{ \left[\lim_{y \rightarrow +\infty} (\mathrm{i}y)^{-1} \right] \left(\lim_{y \rightarrow +\infty} \mathrm{i}y \left[G(\mathrm{i}y) + \sum_{j=0}^0 (\mathrm{i}y)^{-(j+1)} t_j \right] \right) \right. \\ &\quad \left. - \left[\lim_{y \rightarrow +\infty} (\mathrm{i}y)^{-1} \right] t_0 \right\} \\ &= \lim_{y \rightarrow +\infty} \left[I_q + s_0^\dagger G(\mathrm{i}y) \right] = \lim_{y \rightarrow +\infty} N(\mathrm{i}y), \end{aligned} \quad (11.5)$$

where

$$N(z) := I_q + s_0^\dagger G(z) \quad (11.6)$$

for all $z \in \Pi_+$. From (11.2), (11.6), (D.2), (C.1), and Lemma 9.8, for each $y \in (0, +\infty)$, we obtain

$$\det N(\mathrm{i}y) \neq 0 \quad (11.7)$$

and

$$G^{[-, \alpha, s_0]}(\mathrm{i}y) = -(\mathrm{i}y - \alpha)^{-1} s_0 [N(\mathrm{i}y)]^{-1}. \quad (11.8)$$

Because of (11.7) and (11.5), we conclude that

$$\lim_{y \rightarrow +\infty} [N(\mathrm{i}y)]^{-1} = I_q^{-1} = I_q. \quad (11.9)$$

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Using (7.1), we obtain that

$$\begin{aligned}
& (\mathrm{i}y - \alpha) \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \\
&= (\mathrm{i}y)^0 s_0 + \sum_{j=1}^m (\mathrm{i}y)^{-j} s_j - \alpha \left[\sum_{j=0}^{m-1} (\mathrm{i}y)^{-(j+1)} s_j \right] - \alpha (\mathrm{i}y)^{-(m+1)} s_m \\
&= (\mathrm{i}y)^{-0} s_0 + \sum_{k=1}^m (\mathrm{i}y)^{-k} (-\alpha s_{k-1} + s_k) - (\mathrm{i}y)^{-(m+1)} \alpha s_m \\
&= \sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m.
\end{aligned} \tag{11.10}$$

and, taking $s_0^{[+, \alpha]} = s_0$ into account,

$$-s_0 + \sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} = \sum_{j=0}^{m-1} (\mathrm{i}y)^{-(j+1)} s_{j+1}^{[+, \alpha]} \tag{11.11}$$

are fulfilled for each $y \in (0, +\infty)$. Furthermore, for each $y \in (0, +\infty)$, we obtain

$$\begin{aligned}
& - \sum_{j=0}^m \sum_{k=0}^{m-1} (\mathrm{i}y)^{m-j-k} s_j^{[+, \alpha]} s_0^\dagger t_k = - \sum_{r=0}^{2m-1} \left[\sum_{\substack{k=0 \\ 0 \leq r-k \leq m}}^{m-1} (\mathrm{i}y)^{m-r} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \right] \\
&= - \sum_{r=0}^{2m-1} \left[\sum_{\substack{k=0 \\ r \geq k \geq r-m}}^{m-1} (\mathrm{i}y)^{m-r} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \right] = - \sum_{r=0}^{2m-1} \sum_{k=\max\{0, r-m\}}^{\min\{r, m-1\}} (\mathrm{i}y)^{m-r} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \tag{11.12} \\
&= \sum_{r=0}^{m-1} \left[(\mathrm{i}y)^{m-r} \left(- \sum_{k=0}^r s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \right) \right] - \sum_{r=m}^{2m-1} \left[(\mathrm{i}y)^{m-r} \sum_{k=r-m}^{m-1} s_{r-k}^{[+, \alpha]} s_0^\dagger t_k \right].
\end{aligned}$$

From Proposition 8.4 we get $(s_j)_{j=0}^m \in \mathcal{D}_{q \times q, m}$. Hence, [12, Lemma 7.11] provides us $\sum_{l=0}^j s_{j-l}^{[+, \alpha]} s_0^\dagger t_l = s_{j+1}^{[+, \alpha]}$ for all $j \in \mathbb{Z}_{0, m-1}$. Consequently, for each $y \in (0, +\infty)$, we have

$$\sum_{j=0}^{m-1} (\mathrm{i}y)^{m-j} \left(s_{j+1}^{[+, \alpha]} - \sum_{l=0}^j s_{j-l}^{[+, \alpha]} s_0^\dagger t_l \right) = O_{q \times q}. \tag{11.13}$$

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Using (11.8) and (11.7), for each $y \in (0, +\infty)$, we infer

$$\begin{aligned}
& (\mathrm{i}y)^{m+1} \left[G^{[-, \alpha, s_0]}(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] \\
&= (\mathrm{i}y)^{m+1} \left(-(\mathrm{i}y - \alpha)^{-1} s_0 [N(\mathrm{i}y)]^{-1} + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right) \\
&= (\mathrm{i}y - \alpha)^{-1} (\mathrm{i}y)^{m+1} \left(-s_0 + (\mathrm{i}y - \alpha) \left[\sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] N(\mathrm{i}y) \right) [N(\mathrm{i}y)]^{-1}.
\end{aligned} \tag{11.14}$$

From (11.6), (11.10), and (11.11) we get

$$\begin{aligned}
& -s_0 + (\mathrm{i}y - \alpha) \left[\sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] N(\mathrm{i}y) \\
&= -s_0 + (\mathrm{i}y - \alpha) \left[\sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] [I_q + s_0^\dagger G(\mathrm{i}y)] \\
&= -s_0 + \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] \\
&\quad \times \left(I_q + s_0^\dagger \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k - \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right) \\
&= -s_0 + \sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \\
&\quad + \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \\
&\quad \times \left(\left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] - \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right) \\
&= \sum_{j=0}^{m-1} (\mathrm{i}y)^{-(j+1)} s_{j+1}^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \\
&\quad + \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \\
&\quad + \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left[- \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right]
\end{aligned} \tag{11.15}$$

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for all $y \in (0, +\infty)$. In view of (11.12), we get

$$\begin{aligned}
& -(\mathrm{i}y)^{m+1} \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left[\sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \\
&= - \left[\sum_{j=0}^m (\mathrm{i}y)^{m+1-j} s_j^{[+, \alpha]} - \alpha s_m \right] s_0^\dagger \left[\sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \\
&= - \sum_{j=0}^m \sum_{k=0}^{m-1} (\mathrm{i}y)^{m-j-k} s_j^{[+, \alpha]} s_0^\dagger t_k + \alpha s_m s_0^\dagger \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \\
&= \sum_{j=0}^{m-1} \left[(\mathrm{i}y)^{m-j} \left(- \sum_{\ell=0}^j s_{j-\ell}^{[+, \alpha]} s_0^\dagger t_\ell \right) \right] - \sum_{j=m}^{2m-1} \left[(\mathrm{i}y)^{m-j} \sum_{\ell=j-m}^{m-1} s_{j-\ell}^{[+, \alpha]} s_0^\dagger t_\ell \right] \\
&\quad + \alpha s_m s_0^\dagger t_\ell \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k.
\end{aligned} \tag{11.16}$$

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Because of (11.15), (11.16), and (11.13), we obtain

$$\begin{aligned}
& (\mathrm{i}y)^{m+1} \left(-s_0 + (\mathrm{i}y - \alpha) \left[\sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] N(\mathrm{i}y) \right) \\
&= \sum_{j=0}^{m-1} (\mathrm{i}y)^{m-j} s_{j+1}^{[+, \alpha]} - \alpha s_m \\
&\quad + \mathrm{i}y \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left((\mathrm{i}y)^m \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right) \\
&\quad - (\mathrm{i}y)^{m+1} \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left[\sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \\
&= \sum_{j=0}^{m-1} (\mathrm{i}y)^{m-j} s_{j+1}^{[+, \alpha]} \\
&\quad + \mathrm{i}y \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left((\mathrm{i}y)^m \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right) \\
&\quad + \sum_{j=0}^{m-1} \left[(\mathrm{i}y)^{m-j} \left(- \sum_{l=0}^j s_{j-l}^{[+, \alpha]} s_0^\dagger t_l \right) \right] - \sum_{j=m}^{2m-1} \left[(\mathrm{i}y)^{m-j} \sum_{l=j-m}^{m-1} s_{j-l}^{[+, \alpha]} s_0^\dagger t_l \right] \\
&\quad + \alpha s_m s_0^\dagger \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k - \alpha s_m \\
&= \sum_{j=0}^{m-1} (\mathrm{i}y)^{m-j} \left(s_{j+1}^{[+, \alpha]} - \sum_{l=0}^j s_{j-l}^{[+, \alpha]} s_0^\dagger t_l \right) \\
&\quad + \mathrm{i}y \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left((\mathrm{i}y)^m \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right) \\
&\quad - \sum_{j=m}^{2m-1} \left[(\mathrm{i}y)^{m-j} \sum_{l=j-m}^{m-1} s_{j-l}^{[+, \alpha]} s_0^\dagger t_l \right] + \alpha s_m \left[s_0^\dagger \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k - I_q \right] \\
&= \mathrm{i}y \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \left((\mathrm{i}y)^m \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right) \\
&\quad - \sum_{j=m}^{2m-1} \left[(\mathrm{i}y)^{m-j} \sum_{l=j-m}^{m-1} s_{j-l}^{[+, \alpha]} s_0^\dagger t_l \right] + \alpha s_m \left[s_0^\dagger \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k - I_q \right]
\end{aligned} \tag{11.17}$$

11. On the inverse (α, s_0) -Schur-Stieltjes transform for special subclasses of $\mathcal{S}_{q, [\alpha, +\infty)}$

for all $y \in (0, +\infty)$. In view of (11.14) and (11.17), we obtain

$$\begin{aligned}
& (\mathrm{i}y)^{m+1} \left[G^{[-, \alpha, s_0]}(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right] \\
&= \left\{ (\mathrm{i}y - \alpha)^{-1} \mathrm{i}y \left[\sum_{j=0}^m (\mathrm{i}y)^{-j} s_j^{[+, \alpha]} - (\mathrm{i}y)^{-(m+1)} \alpha s_m \right] s_0^\dagger \right. \\
&\quad \times \left((\mathrm{i}y)^m \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right) \\
&\quad - \sum_{j=m}^{2m-1} \left[(\mathrm{i}y - \alpha)^{-1} (\mathrm{i}y)^{m-j} \sum_{\ell=j-m}^{m-1} s_{j-\ell}^{[+, \alpha]} s_0^\dagger t_\ell \right] \\
&\quad \left. + (\mathrm{i}y - \alpha)^{-1} \alpha s_m \left[s_0^\dagger \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k - I_q \right] \right\} [N(\mathrm{i}y)]^{-1}
\end{aligned} \tag{11.18}$$

for all $y \in (0, +\infty)$. Using (11.4), (11.9), and (11.18), we get

$$\begin{aligned}
O_{q \times q} &= O_{q \times q} \cdot I_q \\
&= \left\{ 1 \cdot \left(s_0^{[+, \alpha]} + \sum_{j=1}^m 0 \cdot s_j^{[+, \alpha]} - 0 \cdot \alpha s_m \right) s_0^\dagger \cdot O_{q \times q} \right. \\
&\quad \left. - \sum_{j=m}^{2m-1} \left(0 \cdot \sum_{\ell=j-m}^{m-1} s_{j-\ell}^{[+, \alpha]} s_0^\dagger t_\ell \right) + 0 \cdot \alpha s_m \left(s_0^\dagger \sum_{k=0}^{m-1} 0 \cdot t_k - I_q \right) \right\} \cdot I_q \\
&= \left\{ \left[\lim_{y \rightarrow +\infty} (y - \alpha)^{-1} y \right] \left(s_0^{[+, \alpha]} + \sum_{j=1}^m \left[\lim_{y \rightarrow +\infty} (\mathrm{i}y)^{-j} \right] s_j^{[+, \alpha]} - \left[\lim_{y \rightarrow +\infty} (\mathrm{i}y)^{-(m+1)} \right] \alpha s_m \right) s_0^\dagger \right. \\
&\quad \times \left(\lim_{y \rightarrow +\infty} (\mathrm{i}y)^m \left[G(\mathrm{i}y) + \sum_{k=0}^{m-1} (\mathrm{i}y)^{-(k+1)} t_k \right] \right) \\
&\quad - \sum_{j=m}^{2m-1} \left[\left(\lim_{y \rightarrow +\infty} [(y - \alpha)^{-1} (\mathrm{i}y)^{m-j}] \right) \sum_{\ell=j-m}^{m-1} s_{j-\ell}^{[+, \alpha]} s_0^\dagger t_\ell \right] \\
&\quad \left. + \left[\lim_{y \rightarrow +\infty} (y - \alpha)^{-1} \right] \alpha s_m \left(s_0^\dagger \sum_{k=0}^{m-1} \left[\lim_{y \rightarrow +\infty} (\mathrm{i}y)^{-(k+1)} \right] t_k - I_q \right) \right\} \left(\lim_{y \rightarrow +\infty} [N(\mathrm{i}y)]^{-1} \right) \\
&= \lim_{y \rightarrow +\infty} (\mathrm{i}y)^{m+1} \left[G^{[-, \alpha, s_0]}(\mathrm{i}y) + \sum_{j=0}^m (\mathrm{i}y)^{-(j+1)} s_j \right].
\end{aligned} \tag{11.19}$$

Since $(s_j)_{j=0}^m$ is, in view of Lemma 8.1, a sequence of Hermitian matrices and because of (11.3) and (11.19), the application of Corollary 6.5 yields $G^{[-, \alpha, s_0]} \in \mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$. \square

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q,[\alpha,+\infty)}$

Corollary 11.4. *Let $\alpha \in \mathbb{R}$, let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty,\alpha}^{\geq,e}$ with α -S-transform $(t_j)_{j=0}^\infty$, and let $G \in \mathcal{S}_{\infty,q,[\alpha,+\infty)}[(t_j)_{j=0}^\infty]$. Then the function $G^{[-,\alpha,s_0]}$ belongs to $\mathcal{S}_{\infty,q,[\alpha,+\infty)}[(s_j)_{j=0}^\infty]$.*

Proof. Combine Remarks 5.1 and 7.1 with Theorem 11.3. \square

Corollary 11.5. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ with α -S-transform $(t_j)_{j=0}^{\kappa-1}$. Denote by $\mathcal{S}_{[+;\alpha,s_0]}$ the bijective mapping defined in Corollary 11.2. Then $\mathcal{S}_{[+;\alpha,s_0]}$ generates a bijective correspondence between the sets $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$ and $\mathcal{S}_{\kappa-1,q,[\alpha,+\infty)}[(t_j)_{j=0}^{\kappa-1}]$. The inverse mapping $\mathcal{S}_{[+;\alpha,s_0]}^{-1}$ is given for $G \in \mathcal{S}_{\kappa-1,q,[\alpha,+\infty)}[(t_j)_{j=0}^{\kappa-1}]$ by $\mathcal{S}_{[+;\alpha,s_0]}^{-1}(G) = G^{[-,\alpha,s_0]}$.*

Proof. We will consider the cases $\kappa \in \mathbb{N}$ and $\kappa = \infty$. In any of these cases, we can apply Corollaries 9.12 and 9.14 to verify the shape of the inverse mapping. If κ belongs to \mathbb{N} , then the assertion follows from Theorems 10.3 and 11.3. Finally, if $\kappa = \infty$, then one has to apply Corollaries 10.4 and 11.4. \square

We mention that the investigations in Sections 10 and 11 were influenced to some extent by considerations in Chen/Hu [17]. In particular, [17, Lemma 2.3] played an essential role for the choice of our strategy. This concerns first the development of a Schur-type algorithm for sequences of complex matrices and then the construction of an interrelated Schur-type algorithm for functions belonging to special subclasses of $\mathcal{S}_{q,[\alpha,+\infty)}$. In [5, 17] Chen and Hu treat the truncated matricial Stieltjes moment problem ($\alpha = 0$ in our setting). They introduce a transformation Γ_m via [5, formula (9)] (see also [17, formula (3.2)]) which maps a sequence of length $m+1$ of complex square matrices to a sequence of length m of complex square matrices. This transformation essentially coincides for $\alpha = 0$ and sequences from $\mathcal{D}_{q \times q, m}$ with the first α -Schur-transformation given via (7.2). To describe the respective solution sets, Chen and Hu then reduce the length of the given sequence of prescribed moments in each step by 2, using the transformation $\Gamma_{m-1}\Gamma_m$ (see [5, formula (12)] and [17, formula (3.7)]). To prove [17, Theorem 3.1], they use Lemma 2.6 from their paper [4] on the truncated matricial Hamburger moment problem.

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q,[\alpha,+\infty)}$

The results of Section 10 suggest the construction of a Schur-Nevanlinna-type algorithm for the class $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}$ with $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. The main theme of this section is to work out the details of this algorithm. In Section 10, we fixed an $m \in \mathbb{N}_0$ and a sequence $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$. Theorem 5.3 tells us then that $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m] \neq \emptyset$. Let $F \in \mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$. Then the central theme of Section 10 was to study the (α, s_0) -Schur-Stieltjes transform $F^{[+,\alpha,s_0]}$ of F . The following observation shows that in the case $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ the results of Section 10 can be applied to arbitrary functions belonging to $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}$.

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q, [\alpha, +\infty)}$

Lemma 12.1. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}$ with $[\alpha, +\infty)$ -Stieltjes measure σ_F . Then σ_F belongs to $\mathcal{M}_{\geq, \kappa}^q([\alpha, +\infty))$, the sequence $(s_j)_{j=0}^\kappa$ given by $s_j := \int_{[\alpha, +\infty)} x^j \sigma_F(dx)$ belongs to $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ and F belongs to $\mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$.*

Proof. From (3.5) we see that $\sigma_F \in \mathcal{M}_{\geq, \kappa}^q([\alpha, +\infty))$. Thus, from (5.1) we infer now $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. Hence, Theorem 5.3 yields $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$. \square

In view of Lemma 12.1, we introduce the following construction of $[\alpha, +\infty)$ -Schur-Stieltjes transforms for functions belonging to the class $\mathcal{S}_{\kappa, q, [\alpha, +\infty)}$ with $\kappa \in \mathbb{N}_0 \cup \{\infty\}$.

Definition 12.2. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}$ with $[\alpha, +\infty)$ -Stieltjes measure σ_F . Then the function $F^{[+, \alpha, \sigma_F([\alpha, +\infty))]}$ is called the $[\alpha, +\infty)$ -Schur-Stieltjes transform of F .

Remark 12.3. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. Then $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}$ and $F^{[+, \alpha, s_0]}$ is the $[\alpha, +\infty)$ -Schur-Stieltjes transform of F .

Proposition 12.4. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. Then the $[\alpha, +\infty)$ -Schur-Stieltjes transform of F belongs to $\mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0]$.*

Proof. In view of Remark 12.3, the application of Corollary 10.2 yields the assertion. \square

Lemma 12.5. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $F \in \mathcal{S}_{0, q, [\alpha, +\infty)}$. Furthermore, let G be the $[\alpha, +\infty)$ -Schur-Stieltjes transform of F . Then G belongs to $\mathcal{S}_{m-1, q, [\alpha, +\infty)}$ if and only if F belongs to $\mathcal{S}_{m, q, [\alpha, +\infty)}$.*

Proof. Let σ_F be the $[\alpha, +\infty)$ -Stieltjes measure of F . In view of Definition 12.2, we have

$$G = F^{[+, \alpha, \sigma_F([\alpha, +\infty))]} \quad (12.1)$$

First suppose that

$$G \in \mathcal{S}_{m-1, q, [\alpha, +\infty)}. \quad (12.2)$$

In view of (12.2) and (5.1), then $G \in \mathcal{S}_{0, q, [\alpha, +\infty)}$ and the $[\alpha, +\infty)$ -Stieltjes measure σ_G of G satisfies $\sigma_G \in \mathcal{M}_{q, m-1}^{\geq}([\alpha, +\infty))$. Hence, the matrices

$$t_j := \int_{[\alpha, +\infty)} x^j \sigma_G(dx), \quad j \in \mathbb{Z}_{0, m-1} \quad (12.3)$$

are well defined and, in view of (12.2) and (12.3), we have

$$G \in \mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}]. \quad (12.4)$$

In view of (12.4), then $\mathcal{S}_{m-1, q, [\alpha, +\infty)}[(t_j)_{j=0}^{m-1}] \neq \emptyset$. Thus, Theorem 5.3 implies $(t_j)_{j=0}^{m-1} \in \mathcal{K}_{q, m-1, \alpha}^{\geq, e}$. The matrix $A := \sigma_F([\alpha, +\infty))$ is obviously non-negative Hermitian. Let

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q,[\alpha,+\infty)}$

$(s_j)_{j=0}^m$ be the inverse α -S-transform corresponding to $[(t_j)_{j=0}^{m-1}, A]$. From Definition 7.5 then we get $s_0 = \sigma_F([\alpha, +\infty))$. Using $(t_j)_{j=0}^{m-1} \in \mathcal{K}_{q,m-1,\alpha}^{\geq,e}$ and $A \in \mathbb{C}_{\geq}^{q \times q}$, we conclude from [12, Proposition 10.15] that $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$ and, in particular, $(s_j)_{j=0}^0 \in \mathcal{K}_{q,0,\alpha}^{\geq,e}$. In view of $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}$ and $s_0 = \sigma_F([\alpha, +\infty))$, we infer from Remark 5.7(a) that $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}[(s_j)_{j=0}^0]$. Because of $s_0 = \sigma_F([\alpha, +\infty))$, $F \in \mathcal{S}_{0,q,[\alpha,+\infty)}[(s_j)_{j=0}^0]$, $(s_j)_{j=0}^0 \in \mathcal{K}_{q,0,\alpha}^{\geq,e}$, and Corollary 9.12, we obtain

$$G^{[-,\alpha,s_0]} = \left(F^{[+,\alpha,\sigma_F([\alpha,+\infty))]} \right)^{[-,\alpha,s_0]} = \left(F^{[+,\alpha,s_0]} \right)^{[-,\alpha,s_0]} = F. \quad (12.5)$$

Taking (9.1) and (12.1) into account, we get

$$\mathcal{N}(A) \subseteq \mathcal{N}\left(-A\left(I_q + (i - \alpha)^{-1}[F(i)]^\dagger A\right)\right) = \mathcal{N}\left(F^{[+,\alpha,A]}(i)\right) = \mathcal{N}(G(i)). \quad (12.6)$$

Furthermore, in view of (12.2), we infer from Lemma 3.9 that $\mathcal{N}(G(i)) = \mathcal{N}(\sigma_G([\alpha, +\infty)))$. Because of (12.4), we see from Remark 5.2 that

$$t_0 = \sigma_G([\alpha, +\infty)). \quad (12.7)$$

The combination of (12.6), $\mathcal{N}(G(i)) = \mathcal{N}(\sigma_G([\alpha, +\infty)))$, and (12.7) yields $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$. From (12.1) and (9.1), we conclude that

$$\mathcal{R}(G(i)) = \mathcal{R}\left(F^{[+,\alpha,s_0]}(i)\right) = \mathcal{R}\left(-A\left(I_q + (i - \alpha)^{-1}[F(i)]^\dagger A\right)\right) \subseteq \mathcal{R}(A). \quad (12.8)$$

Taking (12.1) into account, we infer from Lemma 3.9 that $\mathcal{R}(G(i)) = \mathcal{R}(\sigma_G([\alpha, +\infty)))$. In combination with (12.7) and (12.8), this implies $\mathcal{R}(t_0) \subseteq \mathcal{R}(A)$. By virtue of (12.7), (12.5), and [10, Lemma B.2], for each $j \in \mathbb{Z}_{0,m-1}$, we conclude

$$\mathcal{N}(t_0) = \mathcal{N}(\sigma_G([\alpha, +\infty))) \subseteq \mathcal{N}\left(\int_{[\alpha,+\infty)} x^j \sigma_G(dx)\right) = \mathcal{N}(t_j)$$

and

$$\mathcal{R}(t_j) = \mathcal{R}\left(\int_{[\alpha,+\infty)} x^j \sigma_G(dx)\right) \subseteq \mathcal{R}(\sigma_G([\alpha, +\infty))) = \mathcal{R}(t_0).$$

Hence in view of Definition 8.3, the sequence $(t_j)_{j=0}^{m-1}$ belongs to $\mathcal{D}_{q \times q, m-1}$. Taking additionally into account $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$ and $\mathcal{R}(t_0) \subseteq \mathcal{R}(A)$, we obtain from [12, Proposition 10.8] that $(t_j)_{j=0}^{m-1}$ is exactly the α -S-transform $(s_j^{[1,\alpha]})_{j=0}^{m-1}$ of $(s_j)_{j=0}^m$. In view of (12.3), this implies $G \in \mathcal{S}_{m-1,q,[\alpha,+\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}]$. The application of Theorem 11.3 provides us then that $G^{[-,\alpha,s_0]}$ belongs to the class $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$. Thus, (12.4) implies $F \in \mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$ and, consequently, $F \in \mathcal{S}_{m,q,[\alpha,+\infty)}$.

Conversely, now suppose that F belongs to $\mathcal{S}_{m,q,[\alpha,+\infty)}$. For each $j \in \mathbb{Z}_{0,m}$, then the matrix $s_j := \int_{[\alpha,+\infty)} x^j \sigma_F(dx)$ is well defined and F belongs to $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$, which, in view of Theorem 5.3, implies $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$. Thus, if $(t_j)_{j=0}^{m-1}$ denotes the α -S-transform of $(s_j)_{j=0}^m$, then Theorem 10.3 and Remark 12.3 yield that G belongs to $\mathcal{S}_{m-1,q,[\alpha,+\infty)}[(t_j)_{j=0}^{m-1}]$. In particular, G belongs to $\mathcal{S}_{m-1,q,[\alpha,+\infty)}$. \square

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q, [\alpha, +\infty)}$

In view of Lemma 12.5, we introduce in recursive way the following notions, which will play a central role in our further investigations.

Definition 12.6. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}$. We will write $F^{[0, \alpha]}$ for F and call it the 0-th $[\alpha, +\infty)$ -Schur-Stieltjes transform of F . For each $k \in \mathbb{Z}_{1, \kappa+1}$, we will call the $[\alpha, +\infty)$ -Schur-Stieltjes transform $F^{[k, \alpha]}$ of the $(k-1)$ -th $[\alpha, +\infty)$ -Schur-Stieltjes transform $F^{[k-1, \alpha]}$ the k -th $[\alpha, +\infty)$ -Schur-Stieltjes transform of F .

Remark 12.7. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $F \in \mathcal{S}_{\kappa, q, [\alpha, +\infty)}$, and let $k \in \mathbb{Z}_{0, \kappa+1}$. Because of Definition 12.6 and Lemma 12.5, then $F^{[k, \alpha]} \in \mathcal{S}_{\kappa-k, q, [\alpha, +\infty)}$. Furthermore, for each $\ell \in \mathbb{Z}_{0, \kappa-k+1}$, we get $(F^{[k, \alpha]})^{[\ell, \alpha]} = F^{[k+\ell, \alpha]}$.

The content of our next considerations can be described as follows. Let $\kappa \in \mathbb{N} \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$. Then we are going to study the Schur-Stieltjes-type algorithm, which is generated via Definition 12.6, particularly for functions which belong to the class $\mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. For every choice of $F \in \mathcal{S}_{q, [\alpha, +\infty)}$ and $s_0 \in \mathbb{C}^{q \times q}$, we set

$$\mathcal{S}_{[+; \alpha, s_0]}(F) := F^{[+, \alpha, s_0]} \quad \text{and} \quad \mathcal{S}_{[-; \alpha, s_0]}(F) := F^{[-, \alpha, s_0]}.$$

Proposition 12.8. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, and let F belong to $\mathcal{S}_{\kappa, q, [\alpha, +\infty)}[(s_j)_{j=0}^\kappa]$. Further, let $m \in \mathbb{Z}_{0, \kappa}$ and let $F^{[m+1, \alpha]}$ be the $(m+1)$ -th $[\alpha, +\infty)$ -Schur-Stieltjes transform of F . For each $\ell \in \mathbb{Z}_{0, \kappa}$, let $(s_j^{[\ell, \alpha]})_{j=0}^{\kappa-\ell}$ be the ℓ -th α -S-transform of $(s_j)_{j=0}^\kappa$. Then:

$$(a) \quad F^{[m+1, \alpha]} = (\mathcal{S}_{[+; \alpha, s_0^{[m, \alpha]}]} \circ \mathcal{S}_{[+; \alpha, s_0^{[m-1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[+; \alpha, s_0^{[1, \alpha]}]} \circ \mathcal{S}_{[+; \alpha, s_0^{[0, \alpha]}]})(F) \text{ and}$$

$$F^{[m+1, \alpha]} \in \begin{cases} \mathcal{S}_{\kappa-(m+1), q, [\alpha, +\infty)}[(s_j^{[m+1, \alpha]})_{j=0}^{\kappa-(m+1)}] & ; m < \kappa \\ \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0^{[m, \alpha]}] & ; m = \kappa \end{cases}.$$

$$(b) \quad F = (\mathcal{S}_{[-; \alpha, s_0^{[0, \alpha]}]} \circ \mathcal{S}_{[-; \alpha, s_0^{[1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[-; \alpha, s_0^{[m, \alpha]}]})(F^{[m+1, \alpha]}).$$

Proof. (a) In view of the construction of $F^{[m+1, \alpha]}$ (see Definitions 12.6 and 12.2) and Remark 12.3, this follows by induction, using Theorem 10.3 and Corollaries 10.2 and 10.4.

(b) In view of Corollaries 11.2 and 11.5, part (b) is an immediate consequence of (a). \square

Our next considerations are aimed to study the inversion of the Schur-Stieltjes algorithm. The following result can be considered as an inverse statement with respect to Proposition 12.8.

Proposition 12.9. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$. For each $\ell \in \mathbb{Z}_{0, m}$, let $(s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}$ be the ℓ -th α -S-transform of $(s_j)_{j=0}^m$. Further, let $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0^{[m, \alpha]}]$ and let

$$F := (\mathcal{S}_{[-; \alpha, s_0^{[0, \alpha]}]} \circ \mathcal{S}_{[-; \alpha, s_0^{[1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[-; \alpha, s_0^{[m, \alpha]}]})(G).$$

Then the following statements hold true:

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q, [\alpha, +\infty)}$

- (a) $F \in \mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$.
- (b) The $(m + 1)$ -th $[\alpha, +\infty)$ -Schur-Stieltjes transform $F^{[m+1, \alpha]}$ of F satisfies $F^{[m+1, \alpha]} = G$.

Proof. (a) This follows by combining Proposition 11.1 and Theorem 11.3.

(b) Taking the definition of F and (a) into account, the application of Corollaries 11.2 and 11.5 yields

$$G = (\mathcal{S}_{[+; \alpha, s_0^{[m, \alpha]}]} \circ \mathcal{S}_{[+; \alpha, s_0^{[m-1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[+; \alpha, s_0^{[1, \alpha]}]} \circ \mathcal{S}_{[+; \alpha, s_0^{[0, \alpha]}]})(F).$$

Combining this with (a), we infer from part (a) of Proposition 12.8 that $F^{[m+1, \alpha]} = G$. \square

The combination of Propositions 12.8 and 12.9 gives us now a complete description of the Schur-Stieltjes algorithm in the class $\mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$.

Theorem 12.10. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$. For each $\ell \in \mathbb{Z}_{0, m}$, let $(s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}$ be the ℓ -th α -S-transform of $(s_j)_{j=0}^m$. Let*

$$\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]} := \mathcal{S}_{[-; \alpha, s_0^{[0, \alpha]}]} \circ \mathcal{S}_{[-; \alpha, s_0^{[1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[-; \alpha, s_0^{[m, \alpha]}]} \quad (12.9)$$

and let

$$\mathcal{S}_{[+; \alpha, (s_j)_{j=0}^m]} := \mathcal{S}_{[+; \alpha, s_0^{[m, \alpha]}]} \circ \mathcal{S}_{[+; \alpha, s_0^{[m-1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[+; \alpha, s_0^{[1, \alpha]}]} \circ \mathcal{S}_{[+; \alpha, s_0^{[0, \alpha]}]}. \quad (12.10)$$

Then the following statements hold true:

- (a) The mapping $\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]}$ generates a bijective correspondence between the sets $\mathcal{S}_{q, [\alpha, +\infty)}[s_0^{[m, \alpha]}]$ and $\mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$.
- (b) Let $\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]}^{-1}$ be the inverse mapping of $\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]}$. For each matrix-valued function $F \in \mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$, then

$$\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]}^{-1}(F) = \mathcal{S}_{[+; \alpha, (s_j)_{j=0}^m]}(F) = F^{[m+1, \alpha]},$$

where $F^{[m+1, \alpha]}$ is the $(m + 1)$ -th $[\alpha, +\infty)$ -Schur-Stieltjes transform of F .

Proof. Combine Propositions 12.8 and 12.9. \square

Our next considerations are aimed to rewrite the mappings introduced in Theorem 12.10 as linear fractional transformations of matrices. The essential tool in realizing this goal will be the matrix polynomials introduced in Appendix D. More precisely, we will use finite products of such matrix polynomials.

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q, [\alpha, +\infty)}$

Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let $m \in \mathbb{Z}_{0, \kappa}$. For all $\ell \in \mathbb{Z}_{0, m}$, let $(s_j^{[\ell, \alpha]})_{j=0}^{\kappa-\ell}$ be the ℓ -th α -S-transform of $(s_j)_{j=0}^\kappa$. Let the sequence $(V_{\alpha, s_0^{[j, \alpha]}})_j^m$ be given via (D.2). Then, let

$$\mathfrak{V}^{[\alpha, (s_j)_{j=0}^m]} := V_{\alpha, s_0^{[0, \alpha]}} V_{\alpha, s_0^{[1, \alpha]}} \cdots V_{\alpha, s_0^{[m-1, \alpha]}} V_{\alpha, s_0^{[m, \alpha]}} \quad (12.11)$$

and let

$$\mathfrak{V}^{[\alpha, (s_j)_{j=0}^m]} = \begin{bmatrix} \mathfrak{v}_{11}^{[\alpha, (s_j)_{j=0}^m]} & \mathfrak{v}_{12}^{[\alpha, (s_j)_{j=0}^m]} \\ \mathfrak{v}_{21}^{[\alpha, (s_j)_{j=0}^m]} & \mathfrak{v}_{22}^{[\alpha, (s_j)_{j=0}^m]} \end{bmatrix}$$

be the block representation of $\mathfrak{V}^{[\alpha, (s_j)_{j=0}^m]}$ with $p \times p$ block $\mathfrak{v}_{11}^{[\alpha, (s_j)_{j=0}^m]}$.

Remark 12.11. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $m \in \mathbb{Z}_{1, \kappa}$ and all $\ell \in \mathbb{Z}_{0, m-1}$, one can see then from (12.11) and [12, Remark 8.3] that

$$\mathfrak{V}^{[\alpha, (s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}]} = \mathfrak{V}^{[\alpha, (s_j^{[\ell, \alpha]})_{j=0}^{m-(\ell+1)}]} V_{\alpha, s_0^{[m, \alpha]}}$$

and

$$\mathfrak{V}^{[\alpha, (s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}]} = V_{\alpha, s_0^{[\ell, \alpha]}} \mathfrak{V}^{[\alpha, (t_j)_{j=0}^{m-(\ell+1)}]},$$

where $t_j := s_j^{[\ell+1, \alpha]}$ for all $j \in \mathbb{Z}_{0, m-(\ell+1)}$.

Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let $m \in \mathbb{Z}_{0, \kappa}$. For all $\ell \in \mathbb{Z}_{0, m}$, let $(s_j^{[\ell, \alpha]})_{j=0}^{\kappa-\ell}$ be the ℓ -th α -S-transform of $(s_j)_{j=0}^\kappa$. Let the sequence $(W_{\alpha, s_0^{[j, \alpha]}})_j^m$ be given via (D.1). Then, let

$$\mathfrak{W}^{[\alpha, (s_j)_{j=0}^m]} := W_{\alpha, s_0^{[m, \alpha]}} W_{\alpha, s_0^{[m-1, \alpha]}} \cdots W_{\alpha, s_0^{[0, \alpha]}} \quad (12.12)$$

and let

$$\mathfrak{W}^{[\alpha, (s_j)_{j=0}^m]} = \begin{bmatrix} \mathfrak{w}_{11}^{[\alpha, (s_j)_{j=0}^m]} & \mathfrak{w}_{12}^{[\alpha, (s_j)_{j=0}^m]} \\ \mathfrak{w}_{21}^{[\alpha, (s_j)_{j=0}^m]} & \mathfrak{w}_{22}^{[\alpha, (s_j)_{j=0}^m]} \end{bmatrix}$$

be the block representation of $\mathfrak{W}^{[\alpha, (s_j)_{j=0}^m]}$ with $p \times p$ block $\mathfrak{w}_{11}^{[\alpha, (s_j)_{j=0}^m]}$.

Remark 12.12. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $m \in \mathbb{Z}_{1, \kappa}$ and all $\ell \in \mathbb{Z}_{0, m-1}$, one can then see from (12.12) and [12, Remark 8.3] that

$$\mathfrak{W}^{[\alpha, (s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}]} = W_{\alpha, s_0^{[m, \alpha]}} \mathfrak{W}^{[\alpha, (s_j^{[\ell, \alpha]})_{j=0}^{m-(\ell+1)}]}$$

and

$$\mathfrak{W}^{[\alpha, (s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}]} = \mathfrak{W}^{[\alpha, (t_j)_{j=0}^{m-(\ell+1)}]} W_{\alpha, s_0^{[\ell, \alpha]}},$$

where $t_j := s_j^{[\ell+1, \alpha]}$ for all $j \in \mathbb{Z}_{0, m-(\ell+1)}$.

12. A Schur-Stieltjes-type algorithm for the class $\mathcal{S}_{q, [\alpha, +\infty)}$

Proposition 12.13. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$, and let $(s_j^{[m, \alpha]})_{j=0}^0$ be the m -th α -S-transform of $(s_j)_{j=0}^m$. Further, let $G \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0^{[m, \alpha]}]$. Then:*

(a) *For all $z \in \mathbb{C} \setminus [\alpha, +\infty)$,*

$$\det \left[\mathbf{v}_{21}^{[\alpha, (s_j)_{j=0}^m]}(z) G(z) + \mathbf{v}_{22}^{[\alpha, (s_j)_{j=0}^m]}(z) \right] \neq 0.$$

(b) *Let $\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]}$ be given via (12.9). Then*

$$\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]}(G) = \mathcal{S}_{\mathfrak{V}^{[q, q]}_{[\alpha, (s_j)_{j=0}^m]}}(G).$$

Proof. For each $\ell \in \mathbb{Z}_{0, m}$ let $(s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}$ be the ℓ -th α -S-transform of $(s_j)_{j=0}^m$. For each $\ell \in \mathbb{Z}_{0, m}$, in view of Theorem 7.2, then $(s_j^{[\ell, \alpha]})_{j=0}^{m-\ell} \in \mathcal{K}_{q, m-\ell, \alpha}^{\geq, e}$. Thus, for each $\ell \in \mathbb{Z}_{0, m}$, we have

$$(s_j^{[\ell, \alpha]})_{j=0}^0 \in \mathcal{K}_{q, 0, \alpha}^{\geq, e}. \quad (12.13)$$

Because of (12.9), we have

$$\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]} = \mathcal{S}_{[-; \alpha, s_0^{[0, \alpha]}]} \circ \mathcal{S}_{[-; \alpha, s_0^{[1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[-; \alpha, s_0^{[m, \alpha]}]}. \quad (12.14)$$

For each $\ell \in \mathbb{Z}_{0, m}$ we infer from Proposition 11.1 and Theorem 11.3 inductively

$$(\mathcal{S}_{[-; \alpha, s_0^{[\ell, \alpha]}]} \circ \mathcal{S}_{[-; \alpha, s_0^{[\ell+1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[-; \alpha, s_0^{[m, \alpha]}]})(G) \in \mathcal{S}_{m-\ell, q, [\alpha, +\infty)}[(s_j^{[\ell, \alpha]})_{j=0}^{m-\ell}].$$

For each $\ell \in \mathbb{Z}_{0, m}$, then Lemma 5.6 shows that

$$(\mathcal{S}_{[-; \alpha, s_0^{[\ell, \alpha]}]} \circ \mathcal{S}_{[-; \alpha, s_0^{[\ell+1, \alpha]}]} \circ \cdots \circ \mathcal{S}_{[-; \alpha, s_0^{[m, \alpha]}]})(G) \in \mathcal{S}_{q, [\alpha, +\infty)}^\diamond[s_0^{[\ell, \alpha]}]. \quad (12.15)$$

Taking (12.13), (12.14), and (12.15) into account, we get from Proposition 9.9 that

$$\mathcal{S}_{[-; \alpha, (s_j)_{j=0}^m]}(G) = (\mathcal{S}_{V_{\alpha, s_0^{[0, \alpha]}}}^{(q, q)} \circ \mathcal{S}_{V_{\alpha, s_0^{[1, \alpha]}}}^{(q, q)} \circ \cdots \circ \mathcal{S}_{V_{\alpha, s_0^{[m, \alpha]}}}^{(q, q)})(G). \quad (12.16)$$

In view of (12.16) and (12.11), now Proposition C.1 yields (a) and (b). \square

Proposition 12.14. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$, and let F belong to $\mathcal{S}_{m, q, [\alpha, +\infty)}[(s_j)_{j=0}^m]$. Then the following statements hold true:*

(a) *For all $z \in \mathbb{C} \setminus [\alpha, +\infty)$,*

$$\det \left[\mathbf{w}_{21}^{[\alpha, (s_j)_{j=0}^m]}(z) F(z) + \mathbf{w}_{22}^{[\alpha, (s_j)_{j=0}^m]}(z) \right] \neq 0.$$

(b) *Let $\mathcal{S}_{[+; \alpha, (s_j)_{j=0}^m]}$ be given by (12.10). Then*

$$\mathcal{S}_{[+; \alpha, (s_j)_{j=0}^m]}(F) = \mathcal{S}_{\mathfrak{W}^{[q, q]}_{[\alpha, (s_j)_{j=0}^m]}}(F).$$

13. Description of the sets $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$

Proof. For each $\ell \in \mathbb{Z}_{0,m}$ let $(s_j^{[\ell,\alpha]})_{j=0}^{m-\ell}$ be the ℓ -th α -S-transform of $(s_j)_{j=0}^m$. For each $\ell \in \mathbb{Z}_{0,m}$ in view of Theorem 7.2, then

$$(s_j^{[\ell,\alpha]})_{j=0}^{m-\ell} \in \mathcal{K}_{q,m-\ell,\alpha}^{\geq,e}. \quad (12.17)$$

Because of (12.10), we have

$$\mathcal{S}_{[+;\alpha,(s_j)_{j=0}^m]} = \mathcal{S}_{[+;\alpha,s_0^{[m,\alpha]}]} \circ \mathcal{S}_{[+;\alpha,s_0^{[m-1,\alpha]}]} \circ \cdots \circ \mathcal{S}_{[+;\alpha,s_0^{[0,\alpha]}]}. \quad (12.18)$$

For $m = 0$, the assertions of (a) and (b) are an immediate consequence of (12.18), (12.12), and Proposition 9.7. Now let $m \in \mathbb{N}$. For each $\ell \in \mathbb{Z}_{0,m-1}$, we infer from Theorem 10.3 inductively that

$$(\mathcal{S}_{[+;\alpha,s_0^{[\ell,\alpha]}]} \circ \mathcal{S}_{[+;\alpha,s_0^{[\ell-1,\alpha]}]} \circ \cdots \circ \mathcal{S}_{[+;\alpha,s_0^{[0,\alpha]}]})(F) \in \mathcal{S}_{m-\ell,q,[\alpha,+\infty)}[(s_j^{[\ell,\alpha]})_{j=0}^{m-\ell}]. \quad (12.19)$$

Taking (12.17), (12.18), and (12.19) into account, Proposition 9.7 yields

$$\mathcal{S}_{[+;\alpha,(s_j)_{j=0}^m]}(F) = (\mathcal{S}_{W_{\alpha,s_0^{[m,\alpha]}}}^{(q,q)} \circ \mathcal{S}_{W_{\alpha,s_0^{[m-1,\alpha]}}}^{(q,q)} \circ \cdots \circ \mathcal{S}_{W_{\alpha,s_0^{[0,\alpha]}}}^{(q,q)})(F). \quad (12.20)$$

In view of (12.20) and (12.12), now Proposition C.1 yields (a) and (b). \square

13. Description of the sets $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$

Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$. Then we know from Theorem 1.2(a) that the solution set $\mathcal{M}_{q,m}^{\geq}[[\alpha,+\infty);(s_j)_{j=0}^m,=]$ of Problem M $[[\alpha,+\infty);(s_j)_{j=0}^m,=]$ is non-empty. Furthermore, Theorem 5.4 tells us that $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$ is exactly the set of $[\alpha,+\infty)$ -Stieltjes transforms of all the measures belonging to $\mathcal{M}_{q,m}^{\geq}[[\alpha,+\infty);(s_j)_{j=0}^m,=]$. On the basis of our Schur-Stieltjes-type algorithm, which was introduced in Section 12, we have obtained important insights into the structure of the set $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$. Using Theorem 7.4, we will now rewrite the descriptions of $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$, which were obtained in Section 12, in a form which is better adapted to the original data sequence $(s_j)_{j=0}^m$. This is our first main theorem.

Theorem 13.1. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$ with α -Stieltjes parametrization $(Q_j)_{j=0}^m$, and let $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ be defined by (12.11). Then:*

- (a) $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m] = \mathcal{S}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}}^{(q,q)}(\mathcal{S}_{q,[\alpha,+\infty)}^\diamond[Q_m])$.
- (b) *For each $F \in \mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$, there is a unique $G \in \mathcal{S}_{q,[\alpha,+\infty)}^\diamond[Q_m]$ which satisfies*

$$\mathcal{S}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}}^{(q,q)}(G) = F,$$

namely $G = F^{[m+1,\alpha]}$, where $F^{[m+1,\alpha]}$ stands for the $(m+1)$ -th $[\alpha,+\infty)$ -Schur-Stieltjes transform of F .

13. Description of the sets $\mathcal{S}_{\kappa,q,[\alpha,+\infty)}[(s_j)_{j=0}^\kappa]$

Proof. Let $(s_j^{[m,\alpha]})_{j=0}^0$ be the m -th α -S-transform of $(s_j)_{j=0}^m$. In view of Theorem 7.4 and Definition 7.3, then $s_0^{[m,\alpha]} = Q_m$. Hence, the application of Theorem 12.10 and Proposition 12.13 completes the proof. \square

It should be mentioned that a result similar to Theorem 13.1 for $\alpha = 0$ is contained in [16, Theorem 4.1(d)].

Corollary 13.2. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^>$, and let $\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}$ be defined by (12.11). Then*

$$\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m] = \mathcal{S}_{\mathfrak{V}^{[\alpha,(s_j)_{j=0}^m]}}^{(q,q)}(\mathcal{S}_{q,[\alpha,+\infty)}^\diamond).$$

Proof. From Proposition 8.5 we get

$$(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}. \quad (13.1)$$

Because of the assumption $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^>$, we infer from Proposition 8.6 that the matrix Q_m defined in Definition 7.3 is positive Hermitian. In particular, $\det Q_m \neq 0$. Consequently, Remark 4.1 implies $\mathcal{S}_{q,[\alpha,+\infty)}^\diamond[Q_m] = \mathcal{S}_{q,[\alpha,+\infty)}^\diamond$. Thus, taking (13.1) into account, the application of Theorem 13.1(a) completes the proof. \square

Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^>$ with α -Stieltjes parametrization $(Q_j)_{j=0}^m$. Then $(s_j)_{j=0}^m$ is called *completely degenerate*, if $Q_m = O_{q \times q}$. Observe that the set $\mathcal{K}_{q,m,\alpha}^{\geq,\text{cd}}$ of all completely degenerate sequences belonging to $\mathcal{K}_{q,m,\alpha}^{\geq}$ is a subclass of $\mathcal{K}_{q,m,\alpha}^{\geq,e}$ (see [9, Proposition 5.9]). This class is connected to the case of a unique solution, which was already discussed in [16, Theorem 4.1(a)–(c)].

Theorem 13.3. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$. Then:*

(a) *The following statements are equivalent:*

- (i) *The set $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$ consists of exactly one element.*
- (ii) *The sequence $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m,\alpha}^{\geq,\text{cd}}$.*

(b) *If (i) holds true, then $\det \mathbf{v}_{22}^{[\alpha,(s_j)_{j=0}^m]}(z) \neq 0$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$ and*

$$\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m] = \left\{ \mathbf{v}_{12}^{[\alpha,(s_j)_{j=0}^m]}(\mathbf{v}_{22}^{[\alpha,(s_j)_{j=0}^m]})^{-1} \right\}.$$

Proof. “(i) \Rightarrow (ii)”: Use Theorem 3.2, Remark 3.11, (5.1), Theorem 5.4, and [7, Theorems 6.1 and 6.3].

“(ii) \Rightarrow (i)”: Let $(Q_j)_{j=0}^m$ be the α -Stieltjes parametrization of $(s_j)_{j=0}^m$. Then (ii) means $Q_m = O_{q \times q}$. Thus, Proposition 4.7(a), implies that the set $\mathcal{S}_{q,[\alpha,+\infty)}^\diamond[Q_m]$ consists of exactly one element, namely the constant function defined on $\mathbb{C} \setminus [\alpha, +\infty)$ with value $O_{q \times q}$. Using Theorem 13.1(a), then (i) follows, and we see moreover that (b) holds true. \square

A. Some considerations on non-negative Hermitian measures

By using Proposition 4.7, we are able to derive an alternate description of the set $\mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$ if $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$ are arbitrarily given. This gives reformulations of our main result stated in Theorem 13.1.

Theorem 13.4. *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$ with α -Stieltjes parametrization $(Q_j)_{j=0}^m$. Suppose that $r := \text{rank } Q_m$ fulfills $r \geq 1$. Let u_1, u_2, \dots, u_r be an orthonormal basis of $\mathcal{R}(Q_m)$ and let $U := [u_1, u_2, \dots, u_r]$. Then:*

$$(a) \quad \mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m] = \mathcal{S}_{\mathfrak{A}^{[\alpha,(s_j)_{j=0}^m]}^{(q,q)}}^{(q,q)}(\{UfU^* \mid f \in \mathcal{S}_{r,[\alpha,+\infty)}^\diamond\}).$$

(b) Let $F \in \mathcal{S}_{m,q,[\alpha,+\infty)}[(s_j)_{j=0}^m]$. Then there is a unique $f \in \mathcal{S}_{r,[\alpha,+\infty)}^\diamond$ such that

$$\mathcal{S}_{\mathfrak{A}^{[\alpha,(s_j)_{j=0}^m]}^{(q,q)}}^{(q,q)}(UfU^*) = F,$$

namely $f := U^*F^{[m+1,\alpha]}U$, where $F^{[m+1,\alpha]}$ is the $(m+1)$ -th $[\alpha, +\infty)$ -Schur-Stieltjes transform of F .

Proof. Since $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$ implies $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ we get from Proposition 8.2 that $Q_m \in \mathbb{C}_{\geq}^{q \times q}$. In particular, $Q_m^* = Q_m$. Thus, Proposition 4.7(b1) yields

$$\mathcal{S}_{q,[\alpha,+\infty)}^\diamond[Q_m] = \{UfU^* \mid f \in \mathcal{S}_{r,[\alpha,+\infty)}^\diamond\}, \quad (13.2)$$

whereas Remark B.5 implies $U^*U = I_r$.

(a) Because of (13.2), we infer (a) from Theorem 13.1(a).

(b) Using (13.2), $U^*U = I_r$, and (a), we conclude (b) from Theorem 13.1(b). \square

A. Some considerations on non-negative Hermitian measures

In this appendix, we summarize some facts on integration with respect to non-negative Hermitian measures. For a detailed treatment of this subject we refer to Rosenberg [25] (see also Kats [18]). Let (Ω, \mathfrak{A}) be a measurable space. If $\mu = [\mu_{jk}]_{j,k=1}^q$ is a non-negative Hermitian measure on (Ω, \mathfrak{A}) , then each entry function μ_{jk} is a complex measure on (Ω, \mathfrak{A}) . In particular, $\mu_{11}, \mu_{22}, \dots, \mu_{qq}$ are finite non-negative real-valued measures. For each $H \in \mathbb{C}_{\geq}^{q \times q}$, the inequality $H \leq (\text{tr } H)I_q$ holds true. Hence, each entry function μ_{jk} is absolutely continuous with respect to the so-called trace measure $\tau := \sum_{j=1}^q \mu_{jj}$ of μ , i. e., for each $M \in \mathfrak{A}$, which satisfies $\tau(M) = 0$, it follows $\mu(M) = O_{q \times q}$. The Radon-Nikodym derivatives $d\mu_{jk}/d\tau$ are thus well defined up to sets of zero τ -measure. Obviously, the matrix-valued function $\mu'_\tau := [d\mu_{jk}/d\tau]_{j,k=1}^q$ is \mathfrak{A} -measurable and integrable with respect to τ . The matrix-valued function μ'_τ is said to be the trace derivative of μ . If ν is a non-negative real-valued measure on \mathfrak{A} , then let the class of all \mathfrak{A} -measurable $p \times q$ matrix-valued functions $\Phi = [\phi_{jk}]_{j=1,\dots,p}^q$ on Ω such that each ϕ_{jk} is integrable with respect to ν be denoted by $p \times q - \mathcal{L}^1(\Omega, \mathfrak{A}, \nu; \mathbb{C})$. An ordered pair (Φ, Ψ) consisting of an \mathfrak{A} -measurable $p \times q$ matrix-valued function Φ on Ω and an \mathfrak{A} -measurable $r \times q$ matrix-valued function Ψ on Ω is said to be integrable with respect to a non-negative Hermitian

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measure μ on (Ω, \mathfrak{A}) if $\Phi\mu'_\tau\Psi^*$ belongs to $p \times r - \mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})$, where τ is the trace measure of μ . In this case, the integral of (Φ, Ψ) with respect to μ is defined by

$$\int_{\Omega} \Phi d\mu \Psi^* := \int_{\Omega} \Phi \mu'_\tau \Psi^* d\tau$$

and for any $M \in \mathfrak{A}$, the pair $(1_M\Phi, 1_M\Psi)$ is integrable with respect to μ , where 1_M is the indicator function of the set M . Then the integral of (Φ, Ψ) with respect to μ over M is defined by $\int_M \Phi d\mu \Psi^* := \int_{\Omega} (1_M\Phi) \mu'_\tau (1_M\Psi)^* d\tau$. An \mathfrak{A} -measurable complex-valued function f on Ω is said to be integrable with respect to a non-negative Hermitian measure μ on (Ω, \mathfrak{A}) if the pair (fI_q, I_q) is integrable with respect to μ . In this case the integral of f with respect to μ is defined by

$$\int_{\Omega} f d\mu := \int_{\Omega} (fI_q) d\mu I_q^*$$

and for any $M \in \mathfrak{A}$, the function $1_M f$ is integrable with respect to μ . Then the integral of f with respect to μ over M is defined by $\int_M f d\mu := \int_M (fI_q) d\mu I_q^*$. We denote by $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ the set of all \mathfrak{A} -measurable complex-valued functions f on Ω which are integrable with respect to a non-negative Hermitian measure μ on (Ω, \mathfrak{A}) .

We consider a non-negative Hermitian measure μ on (Ω, \mathfrak{A}) with trace measure τ and an \mathfrak{A} -measurable complex-valued function f on Ω :

Remark A.1. The function f belongs to $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ if and only if $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})$.

Remark A.2. Let $u \in \mathbb{C}^q$. Then $\nu := u^* \mu u$ is a finite measure on (Ω, \mathfrak{A}) which is absolutely continuous with respect to τ . If f belongs to $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$, then $\int_{\Omega} |f| d\nu < +\infty$ and $\int_{\Omega} f d\nu = u^* (\int_{\Omega} f d\mu) u$.

Remark A.3. The function f belongs to $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ if and only if $\int_{\Omega} |f| d(u^* \mu u) < +\infty$ for all $u \in \mathbb{C}^q$.

Remark A.4. If $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$, then the functions $\operatorname{Re} f$ and $\operatorname{Im} f$ both belong to $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ and $\int_{\Omega} \operatorname{Re} f d\mu = \operatorname{Re} \int_{\Omega} f d\mu$ and $\int_{\Omega} \operatorname{Im} f d\mu = \operatorname{Im} \int_{\Omega} f d\mu$.

Proposition A.5 (Lebesgue's dominated convergence for non-negative Hermitian measures [13, Proposition A.6]). *Let μ be a non-negative Hermitian $q \times q$ measure on a measurable space (Ω, \mathfrak{A}) with trace measure τ . For all $n \in \mathbb{N}$, let $f_n: \Omega \rightarrow \mathbb{C}$ be an \mathfrak{A} -measurable function. Let $f: \Omega \rightarrow \mathbb{C}$ be an \mathfrak{A} -measurable function and let $g \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ be such that $\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$ for τ -a. a. $\omega \in \Omega$ and that $|f_n(\omega)| \leq |g(\omega)|$ for all $n \in \mathbb{N}$ and τ -a. a. $\omega \in \Omega$. Then $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$, $f_n \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu$.*

B. Some Results On Moore-Penrose Inverses of Matrices

For the convenience of the reader, we state some well-known and some special results on Moore-Penrose inverses of matrices (see e.g., Rao/Mitra [24] or [6]Section 1). If $A \in \mathbb{C}^{p \times q}$, then (by definition) the Moore-Penrose inverse A^\dagger of A is the unique matrix $A^\dagger \in \mathbb{C}^{q \times p}$ which satisfies the four equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$$

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Remark B.1. Let $A \in \mathbb{C}^{p \times q}$. Then one can easily check that:

- (a) $(A^\dagger)^\dagger = A$, $(A^\dagger)^* = (A^*)^\dagger$, and $I_p - AA^\dagger \in \mathbb{C}_{\geq}^{q \times q}$.
- (b) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$, $\text{rank}(A^\dagger) = \text{rank } A$, and $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$.

Proposition B.2 (see, e. g. [6]Theorem 1.1.1). *If $A \in \mathbb{C}^{p \times q}$ then a matrix $G \in \mathbb{C}^{q \times p}$ is the Moore-Penrose inverse of A if and only if $AG = P_A$ and $GA = P_G$, where P_A and P_G are respectively, the matrices associated with the orthogonal projection in \mathbb{C}^p onto $\mathcal{R}(A)$ and the orthogonal projection in \mathbb{C}^q onto $\mathcal{R}(G)$.*

Remark B.3. Let $A \in \mathbb{C}^{p \times q}$. Then it is readily checked that:

- (a) Let $r \in \mathbb{N}$ and let $B \in \mathbb{C}^{r \times q}$. Then $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ if and only if $BA^\dagger A = B$. Furthermore, $\mathcal{N}(B) = \mathcal{N}(A)$ if and only if $B^\dagger B = A^\dagger A$.
- (b) Let $s \in \mathbb{N}$ and let $C \in \mathbb{C}^{p \times s}$. Then $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ if and only if $AA^\dagger C = C$. Furthermore, $\mathcal{R}(C) = \mathcal{R}(A)$ if and only if $CC^\dagger = AA^\dagger$.

Lemma B.4 ([11, Lemma A.5]). *Let $A, X \in \mathbb{C}^{p \times q}$. Then the following statements are equivalent:*

- (i) $\mathcal{N}(A) \subseteq \mathcal{N}(X)$ and $\mathcal{R}(A) \subseteq \mathcal{R}(X)$.
- (ii) $\mathcal{N}(A) = \mathcal{N}(X)$ and $\mathcal{R}(A) = \mathcal{R}(X)$.
- (iii) $A^\dagger A = X^\dagger X$ and $AA^\dagger = XX^\dagger$.
- (iv) $\mathcal{N}(A^\dagger) = \mathcal{N}(X^\dagger)$ and $\mathcal{R}(A^\dagger) = \mathcal{R}(X^\dagger)$.

Remark B.5. Let $A \in \mathbb{C}^{p \times q} \setminus \{O_{p \times q}\}$. Let $r := \text{rank } A$, let u_1, u_2, \dots, u_r be an orthonormal basis of $\mathcal{R}(A^*)$, and let $U := [u_1, u_2, \dots, u_r]$. Then $U^*U = I_r$, and, in view of Proposition B.2 and Remark B.1(b), furthermore $UU^* = A^\dagger A$.

At the end of this section, we give a slight generalization of a result due to S. L. Campbell and C. D. Meyer Jr. This result can be proved by an obvious modification of the proof given in [3, Theorem 10.4.1].

Lemma B.6. *Suppose that $(A_n)_{n=1}^\infty$ is a sequence of complex $p \times q$ matrices which converges to a complex $p \times q$ matrix A . Then $(A_n^\dagger)_{n=1}^\infty$ is convergent if and only if there is a positive integer m such that $\text{rank } A_n = \text{rank } A$ for each integer n with $n \geq m$. In this case, $(A_n^\dagger)_{n=1}^\infty$ converges to A^\dagger .*

C. On Linear Fractional Transformations of Matrices

In this appendix, we summarize some basic facts on linear fractional transformations of matrices which are needed in the paper. This material is mostly taken from [23] and [6, Section 1.6].

D. The Matrix Polynomials $V_{\alpha,A}$ and $W_{\alpha,A}$

Let $a \in \mathbb{C}^{p \times p}$, $b \in \mathbb{C}^{p \times q}$, $c \in \mathbb{C}^{q \times p}$, $d \in \mathbb{C}^{q \times q}$, and let

$$E := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If the set

$$\mathcal{Q}_{[c,d]} := \{x \in \mathbb{C}^{p \times q} \mid \det(cx + d) \neq 0\}$$

is non-empty, then the linear fractional transformation $\mathcal{S}_E^{(p,q)}: \mathcal{Q}_{[c,d]} \rightarrow \mathbb{C}^{p \times q}$ is defined by

$$\mathcal{S}_E^{(p,q)}(x) := (ax + b)(cx + d)^{-1}. \quad (\text{C.1})$$

The following well-known result shows that the composition of two linear fractional transformations is again a mapping of this type.

Proposition C.1 (see, e. g. [6, Proposition 1.6.3]). *Let $a_1, a_2 \in \mathbb{C}^{p \times p}$, let $b_1, b_2 \in \mathbb{C}^{p \times q}$, let $c_1, c_2 \in \mathbb{C}^{q \times p}$, and let $d_1, d_2 \in \mathbb{C}^{q \times q}$ be such that*

$$\text{rank}[c_1, d_1] = \text{rank}[c_2, d_2] = q.$$

Furthermore, let $E_1 := \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $E_2 := \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, $E := E_2 E_1$, and $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the block representation of E with $p \times p$ block a . Then $\mathcal{Q} := \{x \in \mathcal{Q}_{[c_1, d_1]} \mid \mathcal{S}_{E_1}^{(p,q)}(x) \in \mathcal{Q}_{[c_2, d_2]}\}$ is a nonempty subset of the set $\mathcal{Q}_{[c,d]}$ and $\mathcal{S}_{E_2}^{(p,q)}(\mathcal{S}_{E_1}^{(p,q)}(x)) = \mathcal{S}_E^{(p,q)}(x)$ holds true for all $x \in \mathcal{Q}$.

We make the following convention: If a non-empty subset \mathcal{G} of \mathbb{C} and a matrix-valued function $V: \mathcal{G} \rightarrow \mathbb{C}^{2q \times 2q}$ with $q \times q$ block partition $V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ and a matrix-valued function $F: \mathcal{G} \rightarrow \mathbb{C}^{q \times q}$ with $\det[v_{21}(z)F(z) + v_{22}(z)] \neq 0$ for all $z \in \mathcal{G}$ are given, then we will use the notation $\mathcal{S}_V^{(q,q)}(F)$ for the function $\mathcal{S}_V^{(q,q)}(F): \mathcal{G} \rightarrow \mathbb{C}^{q \times q}$ defined by $[\mathcal{S}_V^{(q,q)}(F)](z) := \mathcal{S}_{V(z)}^{(q,q)}(F(z))$ for all $z \in \mathcal{G}$.

D. The Matrix Polynomials $V_{\alpha,A}$ and $W_{\alpha,A}$

In this appendix, we study special linear $(p+q) \times (p+q)$ matrix polynomials which are intensively used in Section 9. Let $\alpha \in \mathbb{C}$ and let $A \in \mathbb{C}^{p \times q}$. Then we define the mappings $W_{\alpha,A}: \mathbb{C} \rightarrow \mathbb{C}^{(p+q) \times (p+q)}$ and $V_{\alpha,A}: \mathbb{C} \rightarrow \mathbb{C}^{(p+q) \times (p+q)}$ by

$$W_{\alpha,A}(z) := \left[\begin{array}{c|c} (z-\alpha)I_p & A \\ \hline -(z-\alpha)A^\dagger & I_q - A^\dagger A \end{array} \right] \quad (\text{D.1})$$

and

$$V_{\alpha,A}(z) := \left[\begin{array}{c|c} O_{p \times p} & -A \\ \hline (z-\alpha)A^\dagger & (z-\alpha)I_q \end{array} \right]. \quad (\text{D.2})$$

The use of the matrix polynomial $V_{\alpha,A}$ was inspired by some constructions in the paper [17]. In particular, we mention [17, formula (2.3)]. In their constructions Chen

D. The Matrix Polynomials $V_{\alpha,A}$ and $W_{\alpha,A}$

and Hu used Drazin inverses instead of Moore-Penrose inverses of matrices. Since both types of generalized inverses coincide for Hermitian matrices (see [14, Proposition A.2]) we can conclude that in the generic case the matrix polynomials $V_{\alpha,A}$ coincide for $\alpha = 0$ with the objects used in [17].

Remark D.1. Let $A \in \mathbb{C}^{p \times q}$ and let $\alpha, z \in \mathbb{C}$. Then one can easily see that

$$V_{\alpha,A}(z)W_{\alpha,A}(z) = (z - \alpha) \operatorname{diag}[AA^\dagger, I_q]$$

and

$$W_{\alpha,A}(z)V_{\alpha,A}(z) = (z - \alpha) \operatorname{diag}[AA^\dagger, I_q].$$

Now we are going to study the linear fractional transformation generated by the matrix $W_{\alpha,A}(z)$ for arbitrarily given $z \in \mathbb{C} \setminus \{\alpha\}$.

Lemma D.2. *Let $\alpha \in \mathbb{C}$, let $A \in \mathbb{C}^{p \times q}$, and let $z \in \mathbb{C} \setminus \{\alpha\}$. Then:*

- (a) *The matrix $-(z - \alpha)^{-1}A$ belongs to $\mathcal{Q}_{[-(z-\alpha)A^\dagger, I_q - A^\dagger A]}$. In particular, $\mathcal{Q}_{[-(z-\alpha)A^\dagger, I_q - A^\dagger A]} \neq \emptyset$.*
- (b) *Let $X \in \mathbb{C}^{p \times q}$ be such that $\mathcal{R}(A) \subseteq \mathcal{R}(X)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(X)$. Then $X \in \mathcal{Q}_{[-(z-\alpha)A^\dagger, I_q - A^\dagger A]}$ and $[-(z - \alpha)A^\dagger X + I_q - A^\dagger A]^{-1} = -(z - \alpha)^{-1}X^\dagger A + I_q - A^\dagger A$.*

Proof. First observe, that $w := z - \alpha \neq 0$.

(a) This follows from $(-wA^\dagger)(-w^{-1}A) + (I_q - A^\dagger A) = I_q$.

(b) In view of Lemma B.4, we have $AA^\dagger = XX^\dagger$ and $A^\dagger A = X^\dagger X$. Hence,

$$\begin{aligned} & (-wA^\dagger X + I_q - A^\dagger A)(-w^{-1}X^\dagger A + I_q - A^\dagger A) \\ &= A^\dagger X X^\dagger A - wA^\dagger X + wA^\dagger X A^\dagger A \\ &\quad - w^{-1}X^\dagger A + I_q - A^\dagger A + w^{-1}A^\dagger A X^\dagger A - A^\dagger A + A^\dagger A A^\dagger A \\ &= A^\dagger A A^\dagger A - wA^\dagger X + wA^\dagger X X^\dagger X \\ &\quad - w^{-1}X^\dagger A + I_q - A^\dagger A + w^{-1}X^\dagger X X^\dagger A - A^\dagger A + A^\dagger A \\ &= A^\dagger A - wA^\dagger X + wA^\dagger X - w^{-1}X^\dagger A + I_q - A^\dagger A + w^{-1}X^\dagger A = I_q. \end{aligned}$$

This completes the proof of (b). □

Lemma D.3. *Let $\alpha \in \mathbb{C}$, let $A \in \mathbb{C}^{p \times q}$, let $W_{\alpha,A}$ be defined via (D.1), and let $X \in \mathbb{C}^{p \times q}$ be such that the inclusions $\mathcal{R}(A) \subseteq \mathcal{R}(X)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(X)$ are satisfied. Furthermore, let $z \in \mathbb{C} \setminus \{\alpha\}$. Then the matrix X belongs to $\mathcal{Q}_{[-(z-\alpha)A^\dagger, I_q - A^\dagger A]}$. Furthermore,*

$$\mathcal{S}_{W_{\alpha,A}(z)}^{(p,q)}(X) = -A[I_q + (z - \alpha)^{-1}X^\dagger A]$$

and

$$\mathcal{R}(\mathcal{S}_{W_{\alpha,A}(z)}^{(p,q)}(X)) \subseteq \mathcal{R}(A), \quad \mathcal{N}(A) \subseteq \mathcal{N}(\mathcal{S}_{W_{\alpha,A}(z)}^{(p,q)}(X)).$$

References

Proof. First observe that $w := z - \alpha \neq 0$. In view of Lemma D.2(b), we have $X \in \mathcal{Q}_{[-wA^\dagger, I_q - A^\dagger A]}$ and

$$(-wA^\dagger X + I_q - A^\dagger A)^{-1} = -w^{-1}X^\dagger A + I_q - A^\dagger A. \quad (\text{D.3})$$

Because of the choice of X , parts (a) and (b) of Remark B.3 yield $XA^\dagger A = X$ and $XX^\dagger A = A$, and, in view of (D.3), then

$$\begin{aligned} \mathcal{S}_{W_{\alpha, A}(z)}^{(p, q)}(X) &= (wX + A)(-wA^\dagger X + I_q - A^\dagger A)^{-1} \\ &= (wX + A)(-w^{-1}X^\dagger A + I_q - A^\dagger A) \\ &= -XX^\dagger A - w^{-1}AX^\dagger A + (wX + A)(I_q - A^\dagger A) \\ &= -A - w^{-1}AX^\dagger A = -A(I_q + w^{-1}X^\dagger A). \end{aligned} \quad (\text{D.4})$$

The remaining assertions are immediate consequences of (D.4). □

Remark D.4. Let $\alpha \in \mathbb{C}$, let $A \in \mathbb{C}^{p \times q}$, let $z \in \mathbb{C} \setminus \{\alpha\}$, and let $X \in \mathcal{Q}_{[(z-\alpha)A^\dagger, (z-\alpha)I_q]}$. Then from (D.2) we see that

$$\mathcal{S}_{V_{\alpha, A}(z)}^{(p, q)}(X) = -(z - \alpha)^{-1}A(I_q + A^\dagger X)^\dagger$$

and, in view of $\det[(z - \alpha)A^\dagger X + (z - \alpha)I_q] \neq 0$, thus $\mathcal{R}(\mathcal{S}_{V_{\alpha, A}(z)}^{(p, q)}(X)) = \mathcal{R}(A)$.

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